

# Game Theory

## 12. Combinatorial Auctions

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# Combinatorial Auctions



## Motivation:

- **Multiple items** are auctioned concurrently.
- Bidders have preferences for **combinations (bundles)** of items.
- Items can **complement** or **substitute** one another.
  - **complement**: left and right shoe together.
  - **substitute**: two right shoes.
- **Aim**: socially optimal **allocation** of items to bidders.



## Applications:

- Spectrum auctions (with combinations of spectrum bands and geographical areas)
- Procurement of transportation services for multiple routes
- ...

## Notation:

- **Items:**  $G = \{1, \dots, m\}$
- **Bidders:**  $N = \{1, \dots, n\}$



## Definition (valuation)

A **valuation** is a function  $v : 2^G \rightarrow \mathbb{R}^+$  with  $v(\emptyset) = 0$  and  $v(S) \leq v(T)$  for  $S \subseteq T \subseteq G$ .

- Requirement  $v(\emptyset) = 0$  to “normalize” valuations.
- Requirement  $v(S) \leq v(T)$  for  $S \subseteq T \subseteq G$ : monotonicity (or “free disposal”).

Let  $S, T \subseteq G$  be disjoint.

- $S$  and  $T$  are **complements** to each other if  $v(S \cup T) > v(S) + v(T)$ .
- $S$  and  $T$  are **substitutes** if  $v(S \cup T) < v(S) + v(T)$ .



## Definition (allocation)

An **allocation** of the items to the bidders is a tuple  $\langle S_1, \dots, S_n \rangle$  with  $S_i \subseteq G$  for  $i = 1, \dots, n$  and  $S_i \cap S_j = \emptyset$  for  $i \neq j$ .

The **social welfare** obtained by an allocation is  $\sum_{i=1}^n v_i(S_i)$  if  $v_1, \dots, v_n$  are the valuations of the bidders.

An allocation is called **socially efficient** if it maximizes social welfare among all allocations.

Let  $A$  be the set of all allocations.

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## Definition (winner determination problem)

Let  $v_i : 2^G \rightarrow \mathbb{R}^+$ ,  $i = 1, \dots, n$ , be the declared valuations of the bidders. The **winner determination problem (WDP)** is the problem of finding a socially efficient allocation  $a \in A$  for these valuations.

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**Aim:** Develop **mechanism** for WDP.

## Challenges:

- Incentive compatibility
- Complexity of representation and communication of preferences (exponentially many subsets of items!)
- Computational complexity



# Single-Minded Bidders

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## Motivation:

- **Focus on single-minded bidders:** cuts complexity of representation down to polynomial space.
- **Idea: single-minded bidder** focuses on one bundle, has fixed valuation  $v^*$  for that bundle (and its supersets), valuation 0 for all other bundles.



## Definition (single-minded bidder)

A valuation  $v$  is called **single-minded** if there is a bundle  $S^* \subseteq G$  and a value  $v^* \in \mathbb{R}^+$  such that

$$v(S) = \begin{cases} v^* & \text{if } S^* \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

A **single-minded bid** is a pair  $\langle S^*, v^* \rangle$ .

- **Representational** complexity: **solved**.
- **Computational** complexity: **not solved**.

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# Allocation Problem for Single-Minded Bidders



## Definition (allocation problem for single-minded bidders)

The **allocation problem for single-minded bidders (APSMB)** is defined by the following input and output.

- **INPUT.** Bids  $\langle S_i^*, v_i^* \rangle$  for  $i = 1, \dots, n$
- **OUTPUT.**  $W \subseteq \{1, \dots, n\}$  with  $S_i^* \cap S_j^* = \emptyset$  for  $i, j \in W, i \neq j$  such that  $\sum_{i \in W} v_i^*$  is maximized.

**Claim:** APSMB is NP-complete.

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# Allocation Problem for Single-Minded Bidders



Since APSMB is an **optimization problem**, consider the corresponding **decision problem**:

**Definition (allocation problem for single-minded bidders, decision problem)**

The **decision problem version of APSMB (APSMB-D)** is defined by the following input and output.

- **INPUT.** Bids  $\langle S_i^*, v_i^* \rangle$  for  $i = 1, \dots, n$  and  $k \in \mathbb{N}$
- **OUTPUT.** Is there a  $W \subseteq \{1, \dots, n\}$  with  $S_i^* \cap S_j^* = \emptyset$  for  $i, j \in W, i \neq j$  such that  $\sum_{i \in W} v_i^* \geq k$ ?

## Theorem

APSMB-D is NP-complete.

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## Proof

NP-hardness: reduction from INDEPENDENT-SET.

INDEPENDENT-SET instance:

- undirected graph  $\langle V, E \rangle$  and  $k_{IS} \in \mathbb{N}$ .
- Question: Is there an independent set of size  $k_{IS}$  in  $\langle V, E \rangle$ ?

Corresponding APSMB-D instance:

- $k = k_{IS}$ , items  $G = E$ , bidders  $N = V$ , and
- for each bidder  $i \in V$  the bid  $\langle S_i^*, v_i^* \rangle$  with  $S_i^* = \{e \in E \mid i \in e\}$  and  $v_i^* = 1$ .
- Question: Is there an allocation with social welfare  $\geq k$ ?
- (Intuitively: Vertices bid for their incident edges.)

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# APSMB-D is NP-complete



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## Proof (ctd.)

Since  $S_i^* \cap S_j^* = \emptyset$  for  $i, j \in W, i \neq j$ , the set of winners  $W$  represents an independent set of cardinality

$$|W| = \sum_{i \in W} v_i^*.$$

Therefore, there is an independent set of cardinality at least  $k_{IS}$  iff there is a set of winners  $W$  with  $\sum_{i \in W} v_i^* \geq k$ . This proves NP-hardness.

APSMB-D  $\in$  NP: obvious (guess and verify set of winners).

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**APSMB-D  $\in$  NP:** obvious (guess and verify set of winners).

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## Consequences:

- Solving APSMB **optimally**: too costly.
- **Alternatives**:
  - **approximation** algorithm
  - **heuristic** approach
  - **special cases**
- **Here**: approximation algorithm

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## Definition (approximation factor)

Let  $c \geq 1$ . An allocation  $\langle S_1, \dots, S_n \rangle$  is a **c-approximation** of an optimal allocation if

$$\sum_{i=1}^n v_i(T_i) \leq c \cdot \sum_{i=1}^n v_i(S_i)$$

for an optimal allocation  $\langle T_1, \dots, T_n \rangle$ .

## Proposition

Approximating APSMB within a factor of  $c \leq m^{1/2-\varepsilon}$  for any  $\varepsilon > 0$  is NP-hard. □

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Best we can still hope for in case of single-minded bidders:

- **incentive compatible**
- **$m^{1/2}$ -approximation algorithm**
- **with polynomial runtime.**

Good news:

- Such an algorithm exists!

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## Definition (mechanism for single-minded bidders)

Let  $V_{sm}$  be the set of all single-minded bids and  $A$  the set of all allocations.

A **mechanism for single-minded bidders** is a tuple

$\langle f, p_1, \dots, p_n \rangle$  consisting of

- a **social choice function**  $f : V_{sm}^n \rightarrow A$  and
- **payment functions**  $p_i : V_{sm}^n \rightarrow \mathbb{R}$  for all  $i = 1, \dots, n$ .

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## Definition (efficient computability)

A mechanism for single-minded bidders is **efficiently computable** if  $f$  and all  $p_i$  can be computed in polynomial time.

## Definition (incentive compatibility)

A mechanism for single-minded bidders is **incentive compatible** if

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i})$$

for all  $i = 1, \dots, n$  and all  $v_1, \dots, v_n, v'_i \in V_{sm}$ , where  $v_i(a) = v_i^*$  if  $i$  wins in  $a$  (gets the desired bundle), and  $v_i(a) = 0$ , otherwise.

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## How to build such a mechanism?

- In principle: could use a **VCG mechanism**.
- Problem with VCG: incentive compatible, but **not efficiently computable**  
(need to compute social welfare, which is NP-hard)
- Alternative idea: VCG-like mechanism that **approximates social welfare**
- Problem with alternative: efficiently computable, but **not incentive compatible**
- Solution: forget VCG, **use specific mechanism for single-minded bidders**.

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# Greedy Mechanism for Single-Minded Bidders



## Definition (greedy mechanism for single-minded bidders)

The **greedy mechanism for single-minded bidders (GMSMB)** is defined as follows.

Let the bidders  $1, \dots, n$  be ordered such that

$$\frac{v_1^*}{\sqrt{|S_1^*|}} \geq \frac{v_2^*}{\sqrt{|S_2^*|}} \geq \dots \geq \frac{v_n^*}{\sqrt{|S_n^*|}}.$$

...

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# Greedy Mechanism for Single-Minded Bidders



Definition (greedy mechanism for single-minded bidders, ctd.)

Let the set  $W \subseteq \{1, \dots, n\}$  be procedurally defined by the following pseudocode:

```
 $W \leftarrow \emptyset$   
for  $i = 1, \dots, n$  do  
  if  $S_i^* \cap \left( \bigcup_{j \in W} S_j^* \right) = \emptyset$  then  
     $W \leftarrow W \cup \{i\}$   
  end if  
end for
```

...

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Definition (greedy mechanism for single-minded bidders, ctd.)

Result: allocation  $a$  where exactly the bidders in  $W$  win.

Payments:

- **Case 1:** If  $i \in W$  and there is a smallest index  $j$  such that  $S_i^* \cap S_j^* \neq \emptyset$  and for all  $k < j$ ,  $k \neq i$ ,  $S_k^* \cap S_j^* = \emptyset$ , then

$$p_i(v_1, \dots, v_n) = \frac{v_j^*}{\sqrt{|S_j^*|/|S_i^*|}},$$

- **Case 2:** Otherwise,

$$p_i(v_1, \dots, v_n) = 0.$$

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## Example

Let  $N = \{1, 2, 3, 4\}$  and  $G = \{1, \dots, 13\}$ .

$i$	Package $S_i^*$	Val. $v_i^*$	$v_i^* / \sqrt{ S_i^* }$	Assignm. order
1	$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$	15		
2	$\{3, 4, 5, 6, 7, 8, 9, 12, 13\}$	3		
3	$\{1, 2, 10, 11\}$	12		
4	$\{10, 11, 12, 13\}$	8		

Positions in assignment order? Winner set? Assignment?  
Social welfare of winner set?

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## Example (ctd.)

### Assignments:

- 1 Bidder 3 gets  $\{1, 2, 10, 11\}$ .
- 2 Bidder 1 gets nothing (obj. 1 and 2 already assigned).
- 3 Bidder 4 gets nothing (obj. 10 and 11 already assigned).
- 4 Bidder 2 gets the remainder, i.e.,  $\{3, 4, 5, 6, 7, 8, 9, 12, 13\}$ .

### Payments:

- 1 Bidder 3 pays

$$\frac{v_1^*}{\sqrt{|S_1^*|/|S_3^*|}} = \frac{15}{\sqrt{9/4}} = \frac{15}{3/2} = 10.$$

- 2 Bidders 1, 4 and 2 pay 0.

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## Example (ctd.)

Therefore:

- Winner set:  $W = \{2, 3\}$ .  
Social welfare:  $U = 12 + 3 = 15$ .
- Optimal winner set:  $W^* = \{1, 4\}$ .  
Optimal social welfare:  $U^* = 15 + 8 = 23$ .
- Approximation ratio:  $23/15 < 2 < 3 < \sqrt{13} = \sqrt{m}$



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Optimal social welfare:  $U^* = 15 + 8 = 23$ .
- Approximation ratio:  $23/15 < 2 < 3 < \sqrt{13} = \sqrt{m}$

# Greedy Mechanism for Single-Minded Bidders: Efficient Computability



## Theorem

GMSMB is efficiently computable.

## Open questions:

- What about incentive compatibility?
- What about approximation factor of  $\sqrt{m}$ ?

# Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility



To prove incentive compatibility:

- **Step 1:** Show that GMSMB is **monotone**.
- **Step 2:** Show that GMSMB **uses critical payments**.
- **Step 3:** Show that in GMSMB **losers pay nothing**.
- **Step 4:** Show that every mechanism for single-minded bidders that is **monotone**, that **uses critical payments**, and where **losers pay nothing** is **incentive compatible**.

# Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility



## Definition (monotonicity)

A mechanism for single-minded bidders is **monotone** if a bidder who wins with bid  $\langle S^*, v^* \rangle$  would also win with any bid  $\langle S', v' \rangle$  where  $S' \subseteq S^*$  and  $v' \geq v^*$  (for fixed bids of the other bidders).

## Definition (critical payments)

A mechanism for single-minded bidders **uses critical payments** if a bidder who wins pays the minimal amount necessary for winning, i.e., the infimum of all  $v'$  such that  $\langle S^*, v' \rangle$  still wins.

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# Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility



## Lemma

GMSBM is monotone, uses critical payments, and losers pay nothing.

## Proof

**Monotonicity:** Increasing  $v_i^*$  or decreasing  $S_i^*$  can only move bidder  $i$  up in the greedy order, making it easier to win.

**Critical payments:** Bidder  $i$  wins as long as he is before bidder  $j$  in the greedy order (if such a  $j$  exists). This holds iff

$$\frac{v_i^*}{\sqrt{|S_i^*|}} \geq \frac{v_j^*}{\sqrt{|S_j^*|}} \quad \text{iff} \quad v_i^* \geq \frac{v_j^* \sqrt{|S_i^*|}}{\sqrt{|S_j^*|}} = \frac{v_j^*}{\sqrt{|S_j^*|/|S_i^*|}} = p_i.$$

Losers pay nothing: Obvious. □

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# Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility



## Lemma

A mechanism for single-minded bidders that is monotone, that uses critical payments, and where losers pay nothing is incentive compatible.

## Proof

(A) Truthful bids never lead to negative utility.

- If the declared bid loses, bidder has utility 0.
- If the declared bid wins, he has utility  $v^* - p^* \geq 0$ , since  $v^* \geq p^*$ , because  $p^*$  is the critical payment, and if the bid wins, the bidder must have (truthfully) bid a value  $v^*$  of at least  $p^*$ .

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# Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility



## Proof (ctd.)

(B) Truthful bids never lead to lower utility than untruthful bids. Suppose declaration of untruthful bid  $\langle S', v' \rangle$  deviating from truthful bid  $\langle S^*, v^* \rangle$ .

(B.1) Case 1: untruthful bid is losing or not useful for bidder.

Suppose  $\langle S', v' \rangle$  is losing or  $S^* \not\subseteq S'$  (bidder does not get the bundle he wants). Then utility  $\leq 0$  in  $\langle S', v' \rangle$ , i.e., no improvement over utility when declaring  $\langle S^*, v^* \rangle$  (cf. (A)).

(B.2) Case 2: untruthful bid is winning and useful for bidder.

Assume  $\langle S', v' \rangle$  is winning and  $S^* \subseteq S'$ . To show that  $\langle S^*, v^* \rangle$  is at least as good a bid as  $\langle S', v' \rangle$ , show that  $\langle S^*, v' \rangle$  is at least as good as  $\langle S', v' \rangle$  and that  $\langle S^*, v^* \rangle$  is at least as good as  $\langle S^*, v' \rangle$ .

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Suppose  $\langle S', v' \rangle$  is losing or  $S^* \not\subseteq S'$  (bidder does not get the bundle he wants). Then utility  $\leq 0$  in  $\langle S', v' \rangle$ , i.e., no improvement over utility when declaring  $\langle S^*, v^* \rangle$  (cf. (A)).

(B.2) Case 2: untruthful bid is winning and useful for bidder.

Assume  $\langle S', v' \rangle$  is winning and  $S^* \subseteq S'$ . To show that  $\langle S^*, v^* \rangle$  is at least as good a bid as  $\langle S', v' \rangle$ , show that  $\langle S^*, v' \rangle$  is at least as good as  $\langle S', v' \rangle$  and that  $\langle S^*, v^* \rangle$  is at least as good as  $\langle S^*, v' \rangle$ .

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## Proof (ctd.)

(B) Truthful bids never lead to lower utility than untruthful bids. Suppose declaration of untruthful bid  $\langle S', v' \rangle$  deviating from truthful bid  $\langle S^*, v^* \rangle$ .

(B.1) Case 1: untruthful bid is losing or not useful for bidder.

Suppose  $\langle S', v' \rangle$  is losing or  $S^* \not\subseteq S'$  (bidder does not get the bundle he wants). Then utility  $\leq 0$  in  $\langle S', v' \rangle$ , i.e., no improvement over utility when declaring  $\langle S^*, v^* \rangle$  (cf. (A)).

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Show that  $\langle S^*, v' \rangle$  is at least as good as  $\langle S', v' \rangle$ .

Let  $p'$  be the payment for bid  $\langle S', v' \rangle$  and  $p$  the payment for bid  $\langle S^*, v' \rangle$ .

For all  $x < p$ ,  $\langle S^*, x \rangle$  is losing, since  $p$  is the critical payment for  $S^*$ .

Due to monotonicity, also  $\langle S', x \rangle$  is losing for all  $x < p$ . Hence, the critical payment  $p'$  for  $S'$  is at least  $p$ .

Thus,  $\langle S^*, v' \rangle$  is still winning, if  $\langle S', v' \rangle$  was, and leads to the same or even lower payment.

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## Proof (ctd.)

- (B.2.b) Lying about valuation does not help.

Show that  $\langle S^*, v^* \rangle$  is at least as good as  $\langle S^*, v' \rangle$ .

- (B.2.b.i) Case 1:  $\langle S^*, v^* \rangle$  is winning with payment  $p^*$ .

If  $v' > p^*$ , then  $\langle S^*, v' \rangle$  is still winning with the same payment, so there is no incentive to deviate to  $\langle S^*, v' \rangle$ .

If  $v' \leq p^*$ , then  $\langle S^*, v' \rangle$  is losing, so there is also no incentive to deviate to  $\langle S^*, v' \rangle$ .

- (B.2.b.ii) Case 2:  $\langle S^*, v^* \rangle$  is losing.

Then  $v^*$  is less than the critical payment, i.e., the payment  $p'$  for a winning bid  $\langle S^*, v' \rangle$  would be greater than  $v^*$ , making a deviation to  $\langle S^*, v' \rangle$  unprofitable.  $\square$

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# Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility



## Corollary

The greedy mechanism for single-minded bidders is incentive compatible.

## Open question:

- What about approximation factor of  $\sqrt{m}$ ?

# Greedy Mechanism for Single-Minded Bidders: Approximation Factor



In the next proof, we will need the **Cauchy-Schwarz inequality**:

## Theorem (Cauchy-Schwarz inequality)

Let  $x_j, y_j \in \mathbb{R}$ . Then

$$\sum_j x_j y_j \leq \sqrt{\sum_j x_j^2} \cdot \sqrt{\sum_j y_j^2}.$$



## Lemma

GMSBM produces a winner set  $W$  that induces a social welfare that is at most a factor  $\sqrt{m}$  worse than the optimal social welfare.

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## Lemma

GMSBM produces a winner set  $W$  that induces a social welfare that is at most a factor  $\sqrt{m}$  worse than the optimal social welfare.

# Greedy Mechanism for Single-Minded Bidders: Approximation Factor



## Proof

- Let  $W^*$  be a set of winning bidders such that  $\sum_{i \in W^*} v_i^*$  is maximal and  $S_i^* \cap S_j^* = \emptyset$  for  $i, j \in W^*, i \neq j$ .
- Let  $W$  be the result of GMSMB.

Show:

$$\sum_{i \in W^*} v_i^* \leq \sqrt{m} \sum_{i \in W} v_i^*.$$

For  $i \in W$  let

$$W_i^* = \{j \in W^* \mid j \geq i \text{ and } S_i^* \cap S_j^* \neq \emptyset\}$$

be the winners in  $W^*$  identical with  $i$  or not contained in  $W$  because of bidder  $i$ . ...

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# Greedy Mechanism for Single-Minded Bidders: Approximation Factor



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# Greedy Mechanism for Single-Minded Bidders: Approximation Factor



## Proof (ctd.)

Since no  $j \in W_i^*$  is before  $i$  in the greedy ordering, for such  $j$ ,

$$v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_j^*|} \quad \text{and, summing over } j \in W_i^*$$

$$\sum_{j \in W_i^*} v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|}. \quad (1)$$

With Cauchy-Schwarz for  $x_j = 1$  and  $y_j = \sqrt{|S_j^*|}$ :

$$\sum_{j \in W_i^*} \sqrt{|S_j^*|} \leq \sqrt{\sum_{j \in W_i^*} 1^2} \sqrt{\sum_{j \in W_i^*} |S_j^*|} = \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}. \quad (2)$$

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## Proof (ctd.)

For all  $j \in W_i^*$ ,  $S_i^* \cap S_j^* \neq \emptyset$ , i.e., there is a  $g(j) \in S_i^* \cap S_j^*$ .

Since  $W^*$  induces an allocation, for all  $j_1, j_2 \in W_i^*$ ,  $j_1 \neq j_2$ ,

$$S_{j_1}^* \cap S_{j_2}^* = \emptyset$$

Hence,

$$(S_i^* \cap S_{j_1}^*) \cap (S_i^* \cap S_{j_2}^*) = \emptyset$$

i.e.,  $g(j_1) \neq g(j_2)$  for  $j_1, j_2 \in W_i^*$  with  $j_1 \neq j_2$ , making  $g$  an injective function from  $W_i^*$  to  $S_i^*$ .

Thus,

$$|W_i^*| \leq |S_i^*|. \quad (3)$$

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# Greedy Mechanism for Single-Minded Bidders: Approximation Factor



## Proof (ctd.)

Since  $W^*$  induces an allocation and  $W_i^* \subseteq W^*$ ,

$$\sum_{j \in W_i^*} |S_j^*| \leq m. \quad (4)$$

...

# Greedy Mechanism for Single-Minded Bidders: Approximation Factor



## Proof (ctd.)

Recall inequalities (1), (2), (3), and (4):

$$\sum_{j \in W_i^*} v_j^* \stackrel{(1)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|}, \quad |W_i^*| \stackrel{(3)}{\leq} |S_i^*|,$$

$$\sum_{j \in W_i^*} \sqrt{|S_j^*|} \stackrel{(2)}{\leq} \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}, \quad \sum_{j \in W_i^*} |S_j^*| \stackrel{(4)}{\leq} m.$$

With these, we get (5):

$$\begin{aligned} \sum_{j \in W_i^*} v_j^* &\stackrel{(1)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|} \stackrel{(2)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|} \\ &\stackrel{(3)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|} \stackrel{(4)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_i^*|} \sqrt{m} = \sqrt{m} v_i^*. \quad \dots \end{aligned}$$

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## Proof (ctd.)

Recall that for  $i \in W$ ,

$$W_i^* = \{j \in W^* \mid j \geq i \text{ and } S_i^* \cap S_j^* \neq \emptyset\}.$$

Let  $j \in W^*$ .

- If  $j \in W$ : then by definition,  $j \in W_j^*$  (assuming, WLOG,  $S_j^* \neq \emptyset$ ).
- If  $j \notin W$ : then there must be some  $i \in W$  such that  $j \geq i$  and  $S_i^* \cap S_j^* \neq \emptyset$ , i.e.,  $j \in W_i^*$ .

Therefore, for each  $j \in W^*$ , there is an  $i \in W$  such that  $j \in W_i^*$ :

$$W^* \subseteq \bigcup_{i \in W} W_i^*. \quad \dots \quad (6)$$

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# Greedy Mechanism for Single-Minded Bidders: Approximation Factor



## Proof (ctd.)

Recall that for  $i \in W$ ,

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## Proof (ctd.)

Recall (5) and (6):

$$\sum_{j \in W_i^*} v_j^* \stackrel{(5)}{\leq} \sqrt{m} v_i^*, \quad W^* \stackrel{(6)}{\subseteq} \bigcup_{i \in W} W_i^*.$$

With these, we finally obtain the desired estimation

$$\sum_{i \in W^*} v_i^* \stackrel{(6)}{\leq} \sum_{i \in W} \sum_{j \in W_i^*} v_j^* \stackrel{(5)}{\leq} \sum_{i \in W} \sqrt{m} v_i^* = \sqrt{m} \sum_{i \in W} v_i^*.$$

Thus, the social welfare of  $W$  differs from the optimal social welfare by a factor of at most  $\sqrt{m}$ . □

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# Greedy Mechanism for Single-Minded Bidders



The following theorem summarizes the results in this chapter:

## Theorem

The greedy mechanism for single-minded bidders is efficiently computable, incentive compatible, and leads to an allocation whose social welfare is a  $\sqrt{m}$ -approximation of the optimal social welfare. □



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- In **combinatorial auctions**, bidders bid for **bundles of items**.
- **Exponential space** needed just to represent and communicate valuations.
- **Therefore:** Focus on **special case of single-minded bidders** (compact representation of valuations).
- **Unfortunately**, still, **optimal allocation NP-hard**.
- **Solution:** **approximate** optimal allocation.
- Polynomial-time approximation possible for approximation factor no better than  $\sqrt{m}$ .
- **Greedy mechanism for single-minded bidders:**
  - achieves  **$\sqrt{m}$ -approximation** of social welfare,
  - is **efficiently computable**, and
  - is **incentive compatible**.