Game Theory

12. Combinatorial Auctions

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1 Combinatorial Auctions

Motivation:

- Multiple items are auctioned concurrently.
- Bidders have preferences for combinations (bundles) of items.
- Items can complement or substitute one another.
  - complement: left and right shoe together.
  - substitute: two right shoes.
- Aim: socially optimal allocation of items to bidders.

Combinatorial Auctions

Applications:

- Spectrum auctions (with combinations of spectrum bands and geographical areas)
- Procurement of transportation services for multiple routes
- ...

Notation:

- Items: $G = \{1, \ldots, m\}$
- Bidders: $N = \{1, \ldots, n\}$

Definition (valuation)

A valuation is a function $v : 2^G \to \mathbb{R}^+$ with $v(\emptyset) = 0$ and $v(S) \leq v(T)$ for $S \subseteq T \subseteq G$.

- Requirement $v(\emptyset) = 0$ to "normalize" valuations.
- Requirement $v(S) \leq v(T)$ for $S \subseteq T \subseteq G$: monotonicity (or "free disposal").

Let $S, T \subseteq G$ be disjoint.

- S and T are complements to each other if $v(S \cup T) > v(S) + v(T)$.
- S and T are substitutes if $v(S \cup T) < v(S) + v(T)$.

Allocations

Definition (allocation)

An allocation of the items to the bidders is a tuple $\langle S_1, \dots, S_n \rangle$ with $S_i \subseteq G$ for $i = 1, \dots, n$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

The social welfare obtained by an allocation is $\sum_{i=1}^{n} v_i(S_i)$ if $v_1, \dots, v_n$ are the valuations of the bidders.

An allocation is called socially efficient if it maximizes social welfare among all allocations.

Let A be the set of all allocations.

Winner Determination Problem

Definition (winner determination problem)

Let $v_i : 2^G \to \mathbb{R}^+$, $i = 1, \ldots, n$, be the declared valuations of the bidders. The winner determination problem (WDP) is the problem of finding a socially efficient allocation $a \in A$ for these valuations.

Aim: Develop mechanism for WDP.

Challenges:

- Incentive compatibility
- Complexity of representation and communication of preferences (exponentially many subsets of items!)
- Computational complexity

2 Single-Minded Bidders

- Definitions
- Complexity
- Greedy Mechanism for Single-Minded Bidders
- Properties of Greedy Mechanism

Special Case: Single-Minded Bidders

Motivation:

- Focus on single-minded bidders: cuts complexity of representation down to polynomial space.
- Idea: single-minded bidder focuses on one bundle, has fixed valuation v^* for that bundle (and its supersets), valuation 0 for all other bundles.

Single-Minded Bidders

Definition (single-minded bidder)

A valuation v is called single-minded if there is a bundle $S^* \subseteq G$ and a value $v^* \in \mathbb{R}^+$ such that

$$v(S) = \begin{cases} v^* & \text{if } S^* \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

A single-minded bid is a pair $\langle S^*, v^* \rangle$.

- Representational complexity: solved.
- Computational complexity: not solved.

Allocation Problem for Single-Minded Bidders

Definition (allocation problem for single-minded bidders)

The allocation problem for single-minded bidders (APSMB) is defined by the following input and output.

- INPUT: Bids $\langle S_i^*, v_i^* \rangle$ for $i = 1, \ldots, n$
- OUTPUT: $W \subseteq \{1, \ldots, n\}$ with $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W$, $i \neq j$ such that $\sum_{i \in W} v_i^*$ is maximized.

Claim: APSMB is NP-complete.

Allocation Problem for Single-Minded Bidders

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Since APSMB is an optimization problem, consider the corresponding decision problem:

Definition (allocation problem for single-minded bidders, decision problem)

The decision problem version of APSMB (APSMB-D) is defined by the following input and output.

- INPUT: Bids $\langle S_i^*, v_i^* \rangle$ for $i = 1, \dots, n$ and $k \in \mathbb{N}$
- OUTPUT: Is there a $W \subseteq \{1, \dots, n\}$ with $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W$, $i \neq j$ such that $\sum_{i \in W} v_i^* \geq k$?

Theorem

APSMB-D is NP-complete.

APSMB-D is NP-complete

Proof

NP-hardness: reduction from INDEPENDENT-SET.

INDEPENDENT-SET instance:

- undirected graph $\langle V, E \rangle$ and $k_{IS} \in \mathbb{N}$.
- Question: Is there an independent set of size k_{IS} in $\langle V, E \rangle$?

Corresponding APSMB-D instance:

- $k = k_{IS}$, items $G = E$, bidders $N = V$, and
- for each bidder $i \in V$ the bid $\langle S_i^*, v_i^* \rangle$ with $S_i^* = \{e \in E \mid i \in e\}$ and $v_i^* = 1$.
- Question: Is there an allocation with social welfare $\geq k$?
- (Intuitively: Vertices bid for their incident edges.)

APSMB-D is NP-complete

Proof (ctd.)

Since $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W$, $i \neq j$, the set of winners W represents an independent set of cardinality

$$|W| = \sum_{i \in W} v_i^*.$$

Therefore, there is an independent set of cardinality at least k_{IS} iff there is a set of winners W with $\sum_{i \in W} v_i^* \geq k$.
This proves NP-hardness.

APSMB-D $\in$ NP: obvious (guess and verify set of winners).

APSMB-D is NP-complete

Consequences:

- Solving APSMB optimally: too costly.
- Alternatives:
  - approximation algorithm
  - heuristic approach
  - special cases
- Here: approximation algorithm

Definition (approximation factor)

Let $c \geq 1$. An allocation $\langle S_1, \ldots, S_n \rangle$ is a *c-approximation* of an optimal allocation if

$$\sum_{i=1}^{n} v_i(T_i) \leq c \cdot \sum_{i=1}^{n} v_i(S_i)$$

for an optimal allocation $\langle T_1, \ldots, T_n \rangle$.

Proposition

Approximating APSMB within a factor of $c \leq m^{1/2-\varepsilon}$ for any $\varepsilon > 0$ is NP-hard. □

Approximation Algorithms

Best we can still hope for in case of single-minded bidders:

- incentive compatible
- $m^{1/2}$-approximation algorithm
- with polynomial runtime.

Good news:

- Such an algorithm exists!

Mechanism for Single-Minded Bidders

Definition (mechanism for single-minded bidders)

Let V_{sm} be the set of all single-minded bids and A the set of all allocations.

A mechanism for single-minded bidders is a tuple $\langle f, p_1, \ldots, p_n \rangle$ consisting of

- a social choice function $f : V_{sm}^n \rightarrow A$ and
- payment functions $p_i : V_{sm}^n \rightarrow \mathbb{R}$ for all $i = 1, \ldots, n$.

Mechanism for Single-Minded Bidders

Definition (efficient computability)

A mechanism for single-minded bidders is efficiently computable if f and all p_i can be computed in polynomial time.

Definition (incentive compatibility)

A mechanism for single-minded bidders is incentive compatible if

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i})$$

for all $i = 1, \ldots, n$ and all $v_1, \ldots, v_n, v_i' \in V_{sm}$, where $v_i(a) = v_i^*$ if i wins in a (gets the desired bundle), and $v_i(a) = 0$, otherwise.

Mechanism for Single-Minded Bidders

How to build such a mechanism?

- In principle: could use a VCG mechanism.
- Problem with VCG: incentive compatible, but not efficiently computable
  (need to compute social welfare, which is NP-hard)
- Alternative idea: VCG-like mechanism that approximates social welfare
- Problem with alternative: efficiently computable, but not incentive compatible
- Solution: forget VCG, use specific mechanism for single-minded bidders.

Greedy Mechanism for Single-Minded Bidders

Definition (greedy mechanism for single-minded bidders)

The greedy mechanism for single-minded bidders (GMSMB) is defined as follows.

Let the bidders $1, \ldots, n$ be ordered such that

$$\frac{v_1^*}{\sqrt{|S_1^*|}} \geq \frac{v_2^*}{\sqrt{|S_2^*|}} \geq \cdots \geq \frac{v_n^*}{\sqrt{|S_n^*|}}.$$

...

Greedy Mechanism for Single-Minded Bidders

Definition (greedy mechanism for single-minded bidders, ctd.)

Let the set $W \subseteq \{1, \ldots, n\}$ be procedurally defined by the following pseudocode:

$W \leftarrow \emptyset$
for $i = 1, \ldots, n$ **do**
  if $S_i^* \cap \left(\bigcup_{j \in W} S_j^* \right) = \emptyset$ **then**
    $W \leftarrow W \cup \{i\}$
  end if
end for

...

Greedy Mechanism for Single-Minded Bidders

Definition (greedy mechanism for single-minded bidders, ctd.)

Result: allocation a where exactly the bidders in W win.

Payments:

- Case 1: If $i \in W$ and there is a smallest index j such that $S_i^* \cap S_j^* \neq \emptyset$ and for all $k < j$, $k \neq i$, $S_k^* \cap S_j^* = \emptyset$, then

$$p_i(v_1, \ldots, v_n) = \frac{v_j^*}{\sqrt{|S_j^*|/|S_i^*|}},$$

- Case 2: Otherwise,

$$p_i(v_1, \ldots, v_n) = 0.$$

Greedy Mechanism for Single-Minded Bidders

Example

Let $N = \{1, 2, 3, 4\}$ and $G = \{1, \ldots, 13\}$.

i	**Package** S_i^*	**Val.** v_i^*	$v_i^* / \sqrt{\lvert S_i^* \rvert}$	**Assignm. order**
1	$\{1,2,3,4,5,6,7,8,9\}$	15		
2	$\{3,4,5,6,7,8,9,12,13\}$	3		
3	$\{1,2,10,11\}$	12		
4	$\{10,11,12,13\}$	8		

Positions in assignment order? Winner set? Assignment?
Social welfare of winner set?

Example (ctd.)

Assignments:

1. Bidder 3 gets $\{1, 2, 10, 11\}$.
2. Bidder 1 gets nothing (obj. 1 and 2 already assigned).
3. Bidder 4 gets nothing (obj. 10 and 11 already assigned).
4. Bidder 2 gets the remainder, i.e., $\{3, 4, 5, 6, 7, 8, 9, 12, 13\}$.

Payments:

1. Bidder 3 pays

$$\frac{v_1^*}{\sqrt{|S_1^*|/|S_3^*|}} = \frac{15}{\sqrt{9/4}} = \frac{15}{3/2} = 10.$$

2. Bidders 1, 4 and 2 pay 0.

Example (ctd.)

Therefore:

- Winner set: $W = \{2,3\}$.
  Social welfare: $U = 12+3 = 15$.
- Optimal winner set: $W^* = \{1,4\}$.
  Optimal social welfare: $U^* = 15+8 = 23$.
- Approximation ratio: $23/15 < 2 < 3 < \sqrt{13} = \sqrt{m}$

Greedy Mechanism for Single-Minded Bidders: Efficient Computability

Theorem

GMSMB is efficiently computable. □

Open questions:

- What about incentive compatibility?
- What about approximation factor of $\sqrt{m}$?

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

To prove incentive compatibility:

- Step 1: Show that GMSMB is monotone.
- Step 2: Show that GMSMB uses critical payments.
- Step 3: Show that in GMSMB losers pay nothing.
- Step 4: Show that every mechanism for single-minded bidders that is monotone, that uses critical payments, and where losers pay nothing is incentive compatible.

Definition (monotonicity)

A mechanism for single-minded bidders is monotone if a bidder who wins with bid $\langle S^*, v^* \rangle$ would also win with any bid $\langle S', v' \rangle$ where $S' \subseteq S^*$ and $v' \geq v^*$ (for fixed bids of the other bidders).

Definition (critical payments)

A mechanism for single-minded bidders uses critical payments if a bidder who wins pays the minimal amount necessary for winning, i.e., the infimum of all v' such that $\langle S^*, v' \rangle$ still wins.

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Lemma

GMSBM is monotone, uses critical payments, and losers pay nothing.

Proof

Monotonicity: Increasing v_i^* or decreasing S_i^* can only move bidder i up in the greedy order, making it easier to win.

Critical payments: Bidder i wins as long as he is before bidder j in the greedy order (if such a j exists). This holds iff

$$\frac{v_i^*}{\sqrt{|S_i^*|}} \geq \frac{v_j^*}{\sqrt{|S_j^*|}} \quad \text{iff} \quad v_i^* \geq \frac{v_j^* \sqrt{|S_i^*|}}{\sqrt{|S_j^*|}} = \frac{v_j^*}{\sqrt{|S_j^*|/|S_i^*|}} = p_i.$$

Losers pay nothing: Obvious. □

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Lemma

A mechanism for single-minded bidders that is monotone, that uses critical payments, and where losers pay nothing is incentive compatible.

Proof

(A) Truthful bids never lead to negative utility.

- If the declared bid loses, bidder has utility 0.
- If the declared bid wins, he has utility $v^* - p^* \geq 0$, since $v^* \geq p^*$, because p^* is the critical payment, and if the bid wins, the bidder must have (truthfully) bid a value v^* of at least p^*.

...

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Proof (ctd.)

(B) Truthful bids never lead to lower utility than untruthful bids.
Suppose declaration of untruthful bid $\langle S', v' \rangle$ deviating from truthful bid $\langle S^*, v^* \rangle$.

(B.1) Case 1: untruthful bid is losing or not useful for bidder.
Suppose $\langle S', v' \rangle$ is losing or $S^* \not\subseteq S'$ (bidder does not get the bundle he wants). Then utility ≤ 0 in $\langle S', v' \rangle$, i.e., no improvement over utility when declaring $\langle S^*, v^* \rangle$ (cf. (A)).

(B.2) Case 2: untruthful bid is winning and useful for bidder.
Assume $\langle S', v' \rangle$ is winning and $S^* \subseteq S'$. To show that $\langle S^*, v^* \rangle$ is at least as good a bid as $\langle S', v' \rangle$, show that $\langle S^*, v' \rangle$ is at least as good as $\langle S', v' \rangle$ and that $\langle S^*, v^* \rangle$ is at least as good as $\langle S^*, v' \rangle$.

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Proof (ctd.)

- (B.2.a) Lying about desired bundle does not help.
  Show that $\langle S^*, v' \rangle$ is at least as good as $\langle S', v' \rangle$.

  Let p' be the payment for bid $\langle S', v' \rangle$ and p the payment for bid $\langle S^*, v' \rangle$.

  For all $x < p$, $\langle S^*, x \rangle$ is losing, since p is the critical payment for S^*.

  Due to monotonicity, also $\langle S', x \rangle$ is losing for all $x < p$. Hence, the critical payment p' for S' is at least p.

  Thus, $\langle S^*, v' \rangle$ is still winning, if $\langle S', v' \rangle$ was, and leads to the same or even lower payment.

...

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Proof (ctd.)

- (B.2.b) Lying about valuation does not help.
  Show that $\langle S^*, v^* \rangle$ is at least as good as $\langle S^*, v' \rangle$.
  - (B.2.b.i) Case 1: $\langle S^*, v^* \rangle$ is winning with payment p^*.
    If $v' > p^*$, then $\langle S^*, v' \rangle$ is still winning with the same payment, so there is no incentive to deviate to $\langle S^*, v' \rangle$.
    If $v' \leq p^*$, then $\langle S^*, v' \rangle$ is losing, so there is also no incentive to deviate to $\langle S^*, v' \rangle$.
  - (B.2.b.ii) Case 2: $\langle S^*, v^* \rangle$ is losing.
    Then v^* is less than the critical payment, i.e., the payment p' for a winning bid $\langle S^*, v' \rangle$ would be greater than v^*, making a deviation to $\langle S^*, v' \rangle$ unprofitable. □

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Corollary

The greedy mechanism for single-minded bidders is incentive compatible. □

Open question:

- What about approximation factor of $\sqrt{m}$?

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

In the next proof, we will need the Cauchy-Schwarz inequality:

Theorem (Cauchy-Schwarz inequality)

Let $x_j, y_j \in \mathbb{R}$. Then

$$\sum_j x_j y_j \le \sqrt{\sum_j x_j^2} \cdot \sqrt{\sum_j y_j^2}.$$

□

Lemma

GMSBM produces a winner set W that induces a social welfare that is at most a factor $\sqrt{m}$ worse than the optimal social welfare.

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof

- Let W^* be a set of winning bidders such that $\sum_{i \in W^*} v_i^*$ is maximal and $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W^*$, $i \neq j$.
- Let W be the result of GMSMB.

Show:

$$\sum_{i \in W^*} v_i^* \leq \sqrt{m} \sum_{i \in W} v_i^*.$$

For $i \in W$ let

$$W_i^* = \{j \in W^* \mid j \geq i \text{ and } S_i^* \cap S_j^* \neq \emptyset\}$$

be the winners in W^* identical with i or not contained in W because of bidder i. ...

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

Since no $j \in W_i^*$ is before i in the greedy ordering, for such j,

$$v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_j^*|} \qquad \text{and, summing over } j \in W_i^*$$

$$\sum_{j \in W_i^*} v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|}. \tag{1}$$

With Cauchy-Schwarz for $x_j = 1$ and $y_j = \sqrt{|S_j^*|}$:

$$\sum_{j \in W_i^*} \sqrt{|S_j^*|} \leq \sqrt{\sum_{j \in W_i^*} 1^2} \sqrt{\sum_{j \in W_i^*} |S_j^*|} = \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}. \tag{2}$$

...

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

For all $j \in W_i^*$, $S_i^* \cap S_j^* \neq \emptyset$, i.e., there is a $g(j) \in S_i^* \cap S_j^*$.

Since W^* induces an allocation, for all $j_1, j_2 \in W_i^*$, $j_1 \neq j_2$,

$$S_{j_1}^* \cap S_{j_2}^* = \emptyset$$

Hence,

$$(S_i^* \cap S_{j_1}^*) \cap (S_i^* \cap S_{j_2}^*) = \emptyset$$

i.e., $g(j_1) \neq g(j_2)$ for $j_1, j_2 \in W_i^*$ with $j_1 \neq j_2$, making g an injective function from W_i^* to S_i^*.
Thus,

$$|W_i^*| \leq |S_i^*|. \tag{3}$$

...

Proof (ctd.)

Since W^* induces an allocation and $W_i^* \subseteq W^*$,

$$\sum_{j \in W_i^*} |S_j^*| \leq m. \tag{4}$$

…

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

Recall inequalities (1), (2), (3), and (4):

$$\sum_{j \in W_i^*} v_j^* \overset{(1)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|}, \qquad |W_i^*| \overset{(3)}{\leq} |S_i^*|,$$

$$\sum_{j \in W_i^*} \sqrt{|S_j^*|} \overset{(2)}{\leq} \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}, \qquad \sum_{j \in W_i^*} |S_j^*| \overset{(4)}{\leq} m.$$

With these, we get (5):

$$\sum_{j \in W_i^*} v_j^* \overset{(1)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|} \overset{(2)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}$$

$$\overset{(3)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|} \overset{(4)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_i^*|} \sqrt{m} = \sqrt{m} v_i^*. \qquad \ldots$$

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

Recall that for $i \in W$,

$$W_i^* = \{j \in W^* \mid j \geq i \text{ and } S_i^* \cap S_j^* \neq \emptyset\}.$$

Let $j \in W^*$.

- If $j \in W$: then by definition, $j \in W_j^*$ (assuming, WLOG, $S_j^* \neq \emptyset$).
- If $j \notin W$: then there must be some $i \in W$ such that $j \geq i$ and $S_i^* \cap S_j^* \neq \emptyset$, i.e., $j \in W_i^*$.

Therefore, for each $j \in W^*$, there is an $i \in W$ such that $j \in W_i^*$:

$$W^* \subseteq \bigcup_{i \in W} W_i^*. \qquad \ldots \tag{6}$$

Proof (ctd.)

Recall (5) and (6):

$$\sum_{j \in W_i^*} v_j^* \overset{(5)}{\leq} \sqrt{m} v_i^*, \qquad W^* \overset{(6)}{\subseteq} \bigcup_{i \in W} W_i^*.$$

With these, we finally obtain the desired estimation

$$\sum_{i \in W^*} v_i^* \overset{(6)}{\leq} \sum_{i \in W} \sum_{j \in W_i^*} v_j^* \overset{(5)}{\leq} \sum_{i \in W} \sqrt{m} v_i^* = \sqrt{m} \sum_{i \in W} v_i^*.$$

Thus, the social welfare of W differs from the optimal social welfare by a factor of at most $\sqrt{m}$. □

Greedy Mechanism for Single-Minded Bidders

The following theorem summarizes the results in this chapter:

Theorem

The greedy mechanism for single-minded bidders is efficiently computable, incentive compatible, and leads to an allocation whose social welfare is a $\sqrt{m}$-approximation of the optimal social welfare. □

3 Summary

Summary

- In combinatorial auctions, bidders bid for bundles of items.
- Exponential space needed just to represent and communicate valuations.
- Therefore: Focus on special case of single-minded bidders (compact representation of valuations).
- Unfortunately, still, optimal allocation NP-hard.
- Solution: approximate optimal allocation.
- Polynomial-time approximation possible for approximation factor no better than $\sqrt{m}$.
- Greedy mechanism for single-minded bidders:
  - achieves $\sqrt{m}$-approximation of social welfare,
  - is efficiently computable, and
  - is incentive compatible.