Game Theory

12. Mechanism Design

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

Bernhard Nebel and Robert Mattmüller

Summer semester 2019

Motivation

- Preference relations $\prec$ contain no information about "by how much" one candidate is preferred.
- Idea: Use money to measure this.
- Use money also for transfers between players "for compensation".

Setting

Formalization:

- Set of alternatives A.
- Set of n players N.
- Valuation functions $v_i : A \to \mathbb{R}$ such that $v_i(a)$ denotes the value player i assigns to alternative a.
- Payment functions specifying amount $p_i \in \mathbb{R}$ that player i pays.
- Utility of player i: $u_i(a) = v_i(a) - p_i$.

Second Price Auctions

Second Price Auctions

Second price auctions:

- There are n players bidding for a single item.
- Player i's private valuations of item: w_i.
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner i pays price p^* and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.

Second Price Auctions

Formally:

- $A = N$
- $v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases}$
- What about payments? Say player i wins:
  - $p^* = 0$ (winner pays nothing): bad idea, players would manipulate and publicly declare values $w_i' \gg w_i$.
  - $p^* = w_i$ (winner pays his valuation): bad idea, players would manipulate and publicly declare values $w_i' = w_i - \varepsilon$.
  - better: $p^* = \max_{j \neq i} w_j$ (winner pays second highest bid).

Vickrey Auction

Definition (Vickrey Auction)

The winner of the Vickrey Auction (aka second price auction) is the player i with the highest declared value w_i. He has to pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

Proposition (Vickrey)

Let i be one of the players and w_i his valuation for the item, u_i his utility if he truthfully declares w_i as his valuation of the item, and u_i' his utility if he falsely declares w_i' as his valuation of the item. Then $u_i \geq u_i'$.

Proof

See
http://en.wikipedia.org/wiki/Vickrey_auction. □

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Incentive Compatible Mechanisms

Incentive Compatible Mechanisms

- Idea: Generalization of Vickrey auctions.
- Preferences modeled as functions $v_i : A \to \mathbb{R}$.
- Let V_i be the space of all such functions for player i.
- Unlike for social choice functions: Not only decide about chosen alternative, but also about payments.

Mechanisms

Definition (Mechanism)

A mechanism $\langle f, p_1, \dots, p_n \rangle$ consists of

- a social choice function $f : V_1 \times \cdots \times V_n \to A$ and
- for each player i, a payment function $p_i : V_1 \times \cdots \times V_n \to \mathbb{R}$.

Definition (Incentive Compatibility)

A mechanism $\langle f, p_1, \dots, p_n \rangle$ is called incentive compatible if for each player $i = 1, \dots, n$, for all preferences $v_1 \in V_1, \dots, v_n \in V_n$ and for each preference $v_i' \in V_i$,

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i}).$$

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VCG Mechanisms

VCG Mechanisms

- If $\langle f, p_1, \ldots, p_n \rangle$ is incentive compatible, truthfully declaring ones preference is a dominant strategy.
- The Vickrey-Clarke-Groves mechanism is an incentive compatible mechanism that maximizes “social welfare”, i.e., the sum over all individual utilities $\sum_{i=1}^{n} v_i(a)$.
- Idea: Reflect other players’ utilities in payment functions, align all players’ incentives with goal of maximizing social welfare.

VCG Mechanisms

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Definition (Vickrey-Clarke-Groves mechanism)

A mechanism $\langle f, p_1, \ldots, p_n \rangle$ is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if

1. $f(v_1, \ldots, v_n) \in \operatorname{argmax}_{a \in A} \sum_{i=1}^{n} v_i(a)$ for all $v_1, \ldots, v_n$ and
2. there are functions $h_1, \ldots, h_n$ with $h_i : V_{-i} \to \mathbb{R}$ such that $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$ for all $i = 1, \ldots, n$ and $v_1, \ldots, v_n$.

Note: $h_i(v_{-i})$ independent of player i's declared preference $\Rightarrow$ $h_i(v_{-i}) = c$ constant from player i's perspective.

Utility of player i $= v_i(f(v_1, \ldots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) - c = \sum_{j=1}^{n} v_j(f(v_1, \ldots, v_n)) - c = \text{social welfare} - c$.

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VCG Mechanisms

Theorem (Vickrey-Clarke-Groves)

Every VCG mechanism is incentive compatible.

Proof

Let i, v_{-i}, v_i and v_i' be given. Show: Declaring true preference v_i dominates declaring false preference v_i'.

Let $a = f(v_i, v_{-i})$ and $a' = f(v_i', v_{-i})$.

$$\text{Utility player } i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v_i' \end{cases}$$

Alternative $a = f(v_i, v_{-i})$ maximizes social welfare

$\Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a')$.

$\Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i})$. $\square$

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Clarke Pivot Rule

- So far: payment functions p_i and functions h_i unspecified.
- One possibility: $h_i(v_{-i}) = 0$ for all h_i and v_{-i}.
  Drawback: Too much money distributed among players (more that necessary).
- Further requirements:
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.

Individual Rationality, Positive Transfers

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Definition (individual rationality)

A mechanism is individually rational if all players always get a nonnegative utility, i.e., if for all $i = 1, \dots, n$ and all $v_1, \dots, v_n$,

$$v_i(f(v_1, \dots, v_n)) - p_i(v_1, \dots, v_n) \geq 0.$$

Definition (positive transfers)

A mechanism has no positive transfers if no player is ever paid money, i.e., for all preferences $v_1, \dots, v_n$,

$$p_i(v_1, \dots, v_n) \geq 0.$$

Clarke Pivot Function

Definition (Clarke pivot function)

The Clarke pivot function is the function

$$h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b).$$

- This leads to payment functions

$$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

  for $a = f(v_1, \ldots, v_n)$.
- Player i pays the difference between what the other players could achieve without him and what they achieve with him.
- Each player internalizes the externalities he causes.

Clarke Pivot Function

Example

- Players $N = \{1,2\}$, alternatives $A = \{a,b\}$.
- Values: $v_1(a) = 10$, $v_1(b) = 2$, $v_2(a) = 9$ and $v_2(b) = 15$.
- Without player 1: b best, since $v_2(b) = 15 > 9 = v_2(a)$.
- With player 1: a best, since $v_1(a) + v_2(a) = 10 + 9 = 19 > 17 = 2 + 15 = v_1(b) + v_2(b)$.
- With player 1, other players (i.e., player 2) lose $v_2(b) - v_2(a) = 6$ units of utility.

$\Rightarrow$ Clarke pivot function $h_1(v_2) = 15$

$\Rightarrow$ payment function

$$p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$

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Clarke Pivot Rule

Lemma (Clarke pivot rule)

A VCG mechanism with Clarke pivot functions has no positive transfers. If $v_i(a) \geq 0$ for all $i = 1, \ldots, n$, $v_i \in V_i$ and $a \in A$, then the mechanism is also individually rational.

Proof

Let $a = f(v_1, \ldots, v_n)$ be the alternative maximizing $\sum_{j=1}^{n} v_j(a)$, and b the alternative maximizing $\sum_{j \neq i} v_j(b)$.

Utility of player i: $u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b)$.

Payment function for i: $p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.

Since b maximizes $\sum_{j \neq i} v_j(b)$: $p_i(v_1, \ldots, v_n) \geq 0$ (no positive transfers).

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Proof (ctd.)

Individual rationality: Since $v_i(b) \geq 0$,

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).$$

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Therefore, the mechanism is also individually rational. □

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Vickrey Auction as a VCG Mechanism

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- a maximizes social welfare $\sum_{i=1}^{n} v_i(a)$ iff a maximizes w_a.
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- Winner pays value of second highest bid:

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Example: Bilateral Trade

- Seller s offers item he values with $0 \leq w_s \leq 1$.
- Potential buyer b values item with $0 \leq w_b \leq 1$.
- Alternatives $A = \{\textit{trade}, \textit{no-trade}\}$.
- Valuations:

$$v_s(\textit{no-trade}) = 0, \qquad v_s(\textit{trade}) = -w_s,$$
$$v_b(\textit{no-trade}) = 0, \qquad v_b(\textit{trade}) = w_b.$$

- VCG mechanism maximizes $v_s(a) + v_b(a)$.
- We have

$$v_s(\textit{trade}) + v_b(\textit{trade}) = w_b - w_s,$$
$$v_s(\textit{no-trade}) + v_b(\textit{no-trade}) = 0$$

i.e., *trade* maximizes social welfare iff $w_b \geq w_s$.

Example: Bilateral Trade (ctd.)

- Requirement: if *no-trade* is chosen, neither player pays anything:

$$p_s(v_s, v_b) = p_b(v_s, v_b) = 0.$$

- To that end, choose Clarke pivot function for buyer:

$$h_b(v_s) = \max_{a \in A} v_s(a).$$

- For seller: Modify Clarke pivot function by an additive constant and set

$$h_s(v_b) = \max_{a \in A} v_b(a) - w_b.$$

Example: Bilateral Trade (ctd.)

- For alternative *no-trade*,

$$p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\textit{no-trade})$$
$$= w_b - w_b - 0 = 0 \quad \text{and}$$
$$p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\textit{no-trade})$$
$$= 0 - 0 = 0.$$

- For alternative *trade*,

$$p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\textit{trade})$$
$$= w_b - w_b - w_b = -w_b \quad \text{and}$$
$$p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\textit{trade})$$
$$= 0 + w_s = w_s.$$

Example: Bilateral Trade (ctd.)

- Because $w_b \geq w_s$, the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.
- Without subsidies, no incentive compatible bilateral trade possible.
- Note: Buyer and seller can exploit the system by colluding.

Example: Public Project

- Project costs C units.
- Each citizen i privately values the project at w_i units.
- Government will undertake project if $\sum_i w_i > C$.
- Alternatives: $A = \{\mathit{project}, \mathit{no\text{-}project}\}$.
- Valuations:

$$v_G(\mathit{project}) = -C, \qquad v_G(\mathit{no\text{-}project}) = 0,$$
$$v_i(\mathit{project}) = w_i, \qquad v_i(\mathit{no\text{-}project}) = 0.$$

- VCG mechanism with Clarke pivot rule: for each citizen i,

$$h_i(v_{-i}) = \max_{a \in A} \left(\sum_{j \neq i} v_j(a) + v_G(a) \right)$$
$$= \begin{cases} \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\ 0, & \text{otherwise.} \end{cases}$$

Example: Public Project (ctd.)

- Citizen i pivotal if $\sum_j w_j > C$ and $\sum_{j \neq i} w_j \leq C$.
- Payment function for citizen i:

$$p_i(v_{1..n}, v_G) = h_i(v_{-i}) - \left(\sum_{j \neq i} v_j(f(v_{1..n}, v_G)) + v_G(f(v_{1..n}, v_G)) \right)$$

  - Case 1: Project undertaken, i pivotal:

$$p_i(v_{1..n}, v_G) = 0 - \left(\sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j$$

  - Case 2: Project undertaken, i not pivotal:

$$p_i(v_{1..n}, v_G) = \left(\sum_{j \neq i} w_j - C \right) - \left(\sum_{j \neq i} w_j - C \right) = 0$$

  - Case 3: Project not undertaken:

$$p_i(v_{1..n}, v_G) = 0$$

Example: Public Project (ctd.)

- I.e., citizen i pays nonzero amount

$$C - \sum_{j \neq i} w_j$$

  only if he is pivotal.
- He pays difference between value of project to fellow citizens and cost C, in general less than w_i.
- Generally,

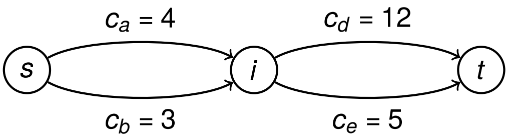
$$\sum_i p_i(project) \leq C$$

  i.e., project has to be subsidized.

Example: Buying a Path in a Network

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player e.
- Each link $e \in E$ has cost c_e if used.
- Objective: procure communication path from s to t.
- Alternatives: $A = \{p \mid p \text{ path from } s \text{ to } t\}$.
- Valuations: $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- Maximizing social welfare:

  minimize $\sum_{e \in p} c_e$ over all paths p from s to t.
- Example:



Example: Buying a Path in a Network (ctd.)

- For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.
- VCG mechanism:

$$h_e(v_{-e}) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'}$$

  i.e., the cost of the cheapest path from s to t in $G \setminus e$.
  (Assume that G is 2-connected, s.t. such p' exists.)
- Payment functions: for chosen path $p = f(v_1, \dots, v_n)$,

$$p_e(v_1, \dots, v_n) = h_e(v_{-e}) - \sum_{e \neq e' \in p} -c_{e'}.$$

  - Case 1: $e \notin p$. Then $p_e(v_1, \dots, v_n) = 0$.
  - Case 2: $e \in p$. Then

$$p_e(v_1, \dots, v_n) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'} - \sum_{e \neq e' \in p} -c_{e'}.$$

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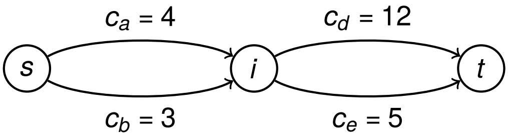

  - Cost along b and e: 8
  - Cost without e: 3
  - Cost of cheapest path without e: 15 (along b and d)
  - Difference is payment: $-15 - (-3) = -12$
    I.e., owner of arc e gets payed 12 for using his arc.

- Note: Alternative path after deletion of e does not necessarily differ from original path at only one position. Could be totally different.

Summary

- New preference model: with money.
- VCG mechanisms generalize Vickrey auctions.
- VCG mechanisms are incentive compatible mechanisms maximizing social welfare.
- With Clarke pivot rule: even no positive transfers and individually rational (if nonnegative valuations).
- Various application areas.