

Game Theory

12. Mechanism Design

Albert-Ludwigs-Universität Freiburg



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- Preference relations \succsim contain no information about “by how much” one candidate is preferred.
- **Idea:** Use **money** to measure this.
- Use money also for **transfers between players** “for compensation”.

Second Price
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Formalization:

- Set of **alternatives** A .
- Set of n **players** N .
- **Valuation functions** $v_i : A \rightarrow \mathbb{R}$ such that $v_i(a)$ denotes the value player i assigns to alternative a .
- **Payment functions** specifying amount $p_i \in \mathbb{R}$ that player i pays.
- **Utility** of player i : $u_i(a) = v_i(a) - p_i$.

1 Second Price Auctions



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Second price auctions:

- There are n players **bidding** for a single item.
- Player i 's **private** valuations of item: w_i .
- **Desired outcome**: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner i pays price p^* and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.

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Formally:

- $A = N$
- $v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases}$
- What about payments? Say player i wins:
 - $p^* = 0$ (winner pays nothing): bad idea, players would manipulate and publicly declare values $w'_i \gg w_i$.
 - $p^* = w_i$ (winner pays his valuation): bad idea, players would manipulate and publicly declare values $w'_i = w_i - \varepsilon$.
 - **better:** $p^* = \max_{j \neq i} w_j$ (winner pays second highest bid).

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Definition (Vickrey Auction)

The winner of the **Vickrey Auction** (aka second price auction) is the player i with the highest declared value w_i . He has to pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

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Proposition (Vickrey)

Let i be one of the players and w_i his valuation for the item, u_i his utility if he truthfully declares w_i as his valuation of the item, and u'_i his utility if he falsely declares w'_i as his valuation of the item. Then $u_i \geq u'_i$.

Proof

See

http://en.wikipedia.org/wiki/Vickrey_auction. □

2 Incentive Compatible Mechanisms



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- **Idea:** Generalization of Vickrey auctions.
- **Preferences** modeled as functions $v_i : A \rightarrow \mathbb{R}$.
- Let V_i be the **space of all such functions** for player i .
- Unlike for social choice functions: Not only decide about **chosen alternative**, but also about **payments**.

Definition (Mechanism)

A **mechanism** $\langle f, p_1, \dots, p_n \rangle$ consists of

- a **social choice function** $f : V_1 \times \dots \times V_n \rightarrow A$ and
- for each player i , a **payment function** $p_i : V_1 \times \dots \times V_n \rightarrow \mathbb{R}$.

Definition (Incentive Compatibility)

A mechanism $\langle f, p_1, \dots, p_n \rangle$ is called **incentive compatible** if for each player $i = 1, \dots, n$, for all preferences $v_1 \in V_1, \dots, v_n \in V_n$ and for each preference $v'_i \in V_i$,

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).$$

3 VCG Mechanisms



- Clarke Pivot Rule
- Examples

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- If $\langle f, p_1, \dots, p_n \rangle$ is **incentive compatible**, **truthfully** declaring ones preference is a **dominant** strategy.
- The **Vickrey-Clarke-Groves mechanism** is an incentive compatible mechanism that maximizes “social welfare”, i.e., the sum over all individual utilities $\sum_{i=1}^n v_i(a)$.
- **Idea**: Reflect other players’ utilities in payment functions, align all players’ incentives with goal of maximizing social welfare.

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Definition (Vickrey-Clarke-Groves mechanism)

A mechanism $\langle f, p_1, \dots, p_n \rangle$ is called a **Vickrey-Clarke-Groves mechanism (VCG mechanism)** if

- 1 $f(v_1, \dots, v_n) \in \operatorname{argmax}_{a \in A} \sum_{i=1}^n v_i(a)$ for all v_1, \dots, v_n and
- 2 there are functions h_1, \dots, h_n with $h_i : V_{-i} \rightarrow \mathbb{R}$ such that $p_i(v_1, \dots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \dots, v_n))$ for all $i = 1, \dots, n$ and v_1, \dots, v_n .

Note: $h_i(v_{-i})$ independent of player i 's declared preference $\Rightarrow h_i(v_{-i}) = c$ constant from player i 's perspective.

Utility of player i $= v_i(f(v_1, \dots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \dots, v_n)) - c = \sum_{j=1}^n v_j(f(v_1, \dots, v_n)) - c = \text{social welfare} - c$.

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Theorem (Vickrey-Clarke-Groves)

Every VCG mechanism is incentive compatible.

Proof

Let i , v_{-i} , v_i and v'_i be given. Show: Declaring true preference v_i dominates declaring false preference v'_i .

Let $a = f(v_i, v_{-i})$ and $a' = f(v'_i, v_{-i})$.

$$\text{Utility player } i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v'_i \end{cases}$$

Alternative $a = f(v_i, v_{-i})$ maximizes social welfare

$$\Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a').$$

$$\Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}). \quad \square$$

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- So far: payment functions p_i and functions h_i unspecified.
- One possibility: $h_i(v_{-i}) = 0$ for all h_i and v_{-i} .
Drawback: Too much money distributed among players (more than necessary).
- Further requirements:
 - Players should pay at most as much as they value the outcome.
 - Players should only pay, never receive money.

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Definition (individual rationality)

A mechanism is **individually rational** if all players always get a nonnegative utility, i.e., if for all $i = 1, \dots, n$ and all v_1, \dots, v_n ,

$$v_i(f(v_1, \dots, v_n)) - p_i(v_1, \dots, v_n) \geq 0.$$

Definition (positive transfers)

A mechanism has **no positive transfers** if no player is ever paid money, i.e., for all preferences v_1, \dots, v_n ,

$$p_i(v_1, \dots, v_n) \geq 0.$$

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Definition (Clarke pivot function)

The **Clarke pivot function** is the function

$$h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b).$$

- This leads to **payment functions**

$$p_i(v_1, \dots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

for $a = f(v_1, \dots, v_n)$.

- Player i pays the difference between what the other players could achieve without him and what they achieve with him.
- Each player **internalizes the externalities** he causes.

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Example

- Players $N = \{1, 2\}$, alternatives $A = \{a, b\}$.
- Values: $v_1(a) = 10$, $v_1(b) = 2$, $v_2(a) = 9$ and $v_2(b) = 15$.
- Without player 1: b best, since $v_2(b) = 15 > 9 = v_2(a)$.
- With player 1: a best, since
 $v_1(a) + v_2(a) = 10 + 9 = 19 > 17 = 2 + 15 = v_1(b) + v_2(b)$.
- With player 1, other players (i.e., player 2) lose
 $v_2(b) - v_2(a) = 6$ units of utility.

⇒ Clarke pivot function $h_1(v_2) = 15$

⇒ payment function

$$p_1(v_1, \dots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$

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Lemma (Clarke pivot rule)

A VCG mechanism with Clarke pivot functions has no positive transfers. If $v_i(a) \geq 0$ for all $i = 1, \dots, n$, $v_i \in V_i$ and $a \in A$, then the mechanism is also individually rational.

Proof

Let $a = f(v_1, \dots, v_n)$ be the alternative maximizing $\sum_{j=1}^n v_j(a)$, and b the alternative maximizing $\sum_{j \neq i} v_j(b)$.

Utility of player i : $u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b)$.

Payment function for i : $p_i(v_1, \dots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.

Since b maximizes $\sum_{j \neq i} v_j(b)$: $p_i(v_1, \dots, v_n) \geq 0$
(no positive transfers).

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Proof (ctd.)

Individual rationality: Since $v_i(b) \geq 0$,

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^n v_j(a) - \sum_{j=1}^n v_j(b).$$

Since a maximizes $\sum_{j=1}^n v_j(a)$,

$$\sum_{j=1}^n v_j(a) \geq \sum_{j=1}^n v_j(b)$$

and hence $u_i \geq 0$.

Therefore, the mechanism is also individually rational. □

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Vickrey Auction as a VCG Mechanism



- $A = N$. Valuations: w_i . $v_a(a) = w_a$, $v_i(a) = 0$ ($i \neq a$).
- a maximizes social welfare $\sum_{i=1}^n v_i(a)$ iff a maximizes w_a .
- Let $a = f(v_1, \dots, v_n) = \operatorname{argmax}_{j \in A} w_j$ be the highest bidder.
- Payments: $p_i(v_1, \dots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.
- But $\max_{b \in A} \sum_{j \neq i} v_j(b) = \max_{b \in A \setminus \{i\}} w_b$.
- **Winner pays value of second highest bid:**

$$\begin{aligned} p_a(v_1, \dots, v_n) &= \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a) \\ &= \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b. \end{aligned}$$

- **Non-winners pay nothing:** For $i \neq a$,

$$\begin{aligned} p_i(v_1, \dots, v_n) &= \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \\ &= \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0. \end{aligned}$$

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Example: Bilateral Trade

- Seller s offers item he values with $0 \leq w_s \leq 1$.
- Potential buyer b values item with $0 \leq w_b \leq 1$.
- Alternatives $A = \{\textit{trade}, \textit{no-trade}\}$.
- Valuations:

$$v_s(\textit{no-trade}) = 0, \quad v_s(\textit{trade}) = -w_s,$$

$$v_b(\textit{no-trade}) = 0, \quad v_b(\textit{trade}) = w_b.$$

- VCG mechanism maximizes $v_s(a) + v_b(a)$.
- We have

$$v_s(\textit{trade}) + v_b(\textit{trade}) = w_b - w_s,$$

$$v_s(\textit{no-trade}) + v_b(\textit{no-trade}) = 0$$

i.e., *trade* maximizes social welfare iff $w_b \geq w_s$.

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Example: Bilateral Trade (ctd.)



- **Requirement:** if *no-trade* is chosen, neither player pays anything:

$$p_s(v_s, v_b) = p_b(v_s, v_b) = 0.$$

- To that end, choose Clarke pivot function **for buyer:**

$$h_b(v_s) = \max_{a \in A} v_s(a).$$

- **For seller:** Modify Clarke pivot function by an additive constant and set

$$h_s(v_b) = \max_{a \in A} v_b(a) - w_b.$$

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Example: Bilateral Trade (ctd.)

- For alternative *no-trade*,

$$p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{no-trade})$$

$$= w_b - w_b - 0 = 0 \quad \text{and}$$

$$p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{no-trade})$$

$$= 0 - 0 = 0.$$

- For alternative *trade*,

$$p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{trade})$$

$$= w_b - w_b - w_b = -w_b \quad \text{and}$$

$$p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{trade})$$

$$= 0 + w_s = w_s.$$

Example: Bilateral Trade (ctd.)



- Because $w_b \geq w_s$, the seller gets at least as much as the buyer pays, i.e., the mechanism **subsidizes** the trade.
- Without subsidies, no incentive compatible bilateral trade possible.
- **Note:** Buyer and seller can exploit the system by **colluding**.

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Example: Public Project

- Project costs C units.
- Each citizen i privately values the project at w_i units.
- Government will undertake project if $\sum_i w_i > C$.
- Alternatives: $A = \{\text{project}, \text{no-project}\}$.
- Valuations:

$$\begin{aligned}v_G(\text{project}) &= -C, & v_G(\text{no-project}) &= 0, \\v_i(\text{project}) &= w_i, & v_i(\text{no-project}) &= 0.\end{aligned}$$

- VCG mechanism with Clarke pivot rule: for each citizen i ,

$$\begin{aligned}h_i(v_{-i}) &= \max_{a \in A} \left(\sum_{j \neq i} v_j(a) + v_G(a) \right) \\ &= \begin{cases} \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

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Example: Public Project (ctd.)

- Citizen i **pivotal** if $\sum_j w_j > C$ and $\sum_{j \neq i} w_j \leq C$.
- **Payment function for citizen i :**

$$p_i(v_{1..n}, v_G) = h_i(v_{-i}) - \left(\sum_{j \neq i} v_j(f(v_{1..n}, v_G)) + v_G(f(v_{1..n}, v_G)) \right)$$

- **Case 1: Project undertaken, i pivotal:**

$$p_i(v_{1..n}, v_G) = 0 - \left(\sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j$$

- **Case 2: Project undertaken, i not pivotal:**

$$p_i(v_{1..n}, v_G) = \left(\sum_{j \neq i} w_j - C \right) - \left(\sum_{j \neq i} w_j - C \right) = 0$$

- **Case 3: Project not undertaken:**

$$p_i(v_{1..n}, v_G) = 0$$

Example: Public Project (ctd.)

- I.e., citizen i pays nonzero amount

$$C - \sum_{j \neq i} w_j$$

only if he is pivotal.

- He pays difference between value of project to fellow citizens and cost C , in general less than w_i .
- Generally,

$$\sum_i p_i(\text{project}) \leq C$$

i.e., project has to be subsidized.

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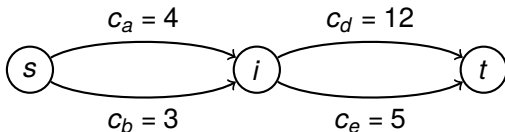
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Example: Buying a Path in a Network



- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player e .
- Each link $e \in E$ has cost c_e if used.
- **Objective:** procure communication path from s to t .
- **Alternatives:** $A = \{p \mid p \text{ path from } s \text{ to } t\}$.
- **Valuations:** $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- **Maximizing social welfare:**
minimize $\sum_{e \in p} c_e$ over all paths p from s to t .
- **Example:**



Example: Buying a Path in a Network (ctd.)



- For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.
- **VCG mechanism:**

$$h_e(v_{-e}) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'}$$

i.e., the cost of the cheapest path from s to t in $G \setminus e$.
(Assume that G is 2-connected, s.t. such p' exists.)

- **Payment functions:** for chosen path $p = f(v_1, \dots, v_n)$,

$$p_e(v_1, \dots, v_n) = h_e(v_{-e}) - \sum_{e' \neq e \in p} -c_{e'}$$

- **Case 1: $e \notin p$.** Then $p_e(v_1, \dots, v_n) = 0$.
- **Case 2: $e \in p$.** Then

$$p_e(v_1, \dots, v_n) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'} - \sum_{e' \neq e \in p} -c_{e'}$$

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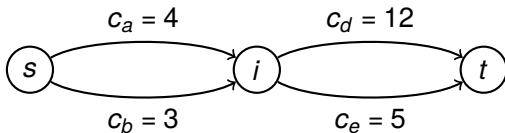
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Example: Buying a Path in a Network (ctd.)



■ Example:



- Cost along b and e : 8
 - Cost without e : 3
 - Cost of cheapest path without e : 15 (along b and d)
 - Difference is payment: $-15 - (-3) = -12$
I.e., owner of arc e gets paid 12 for using his arc.
- **Note:** Alternative path after deletion of e does not necessarily differ from original path at only one position. Could be totally different.

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- New preference model: with **money**.
- VCG mechanisms generalize **Vickrey auctions**.
- **VCG mechanisms** are **incentive compatible** mechanisms maximizing social welfare.
- With **Clarke pivot rule**: even **no positive transfers** and **individually rational** (if nonnegative valuations).
- Various application areas.

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