

# Game Theory

## 6. Extensive Games

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**UNI  
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# 1 Motivation



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- **So far:** All players move **simultaneously**, and then the outcome is determined.
- **Often in practice:** Several moves in **sequence** (e. g. in chess).  
↪ cannot be directly reflected by strategic games.
- **Extensive games** (with perfect information) reflect such situations by modeling games as **game trees**.
- **Idea:** Players have several decision points where they can decide how to play.
- **Strategies:** Mappings from decision points in the game tree to actions to be played.

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# 2 Definitions



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## Definition (Extensive game with perfect information)

An **extensive game with perfect information** is a tuple

$\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  that consists of:

- A finite non-empty set  $N$  of **players**.
- A set  $H$  of (finite or infinite) sequences, called **histories**, such that
  - it contains the empty sequence  $\langle \rangle \in H$ ,
  - $H$  is **closed under prefixes**: if  $\langle a^1, \dots, a^k \rangle \in H$  for some  $k \in \mathbb{N} \cup \{\infty\}$ , and  $l < k$ , then also  $\langle a^1, \dots, a^l \rangle \in H$ , and
  - $H$  is **closed under limits**: if for some infinite sequence  $\langle a^i \rangle_{i=1}^\infty$ , we have  $\langle a^i \rangle_{i=1}^k \in H$  for all  $k \in \mathbb{N}$ , then  $\langle a^i \rangle_{i=1}^\infty \in H$ .

All infinite histories and all histories  $\langle a^i \rangle_{i=1}^k \in H$ , for which there is no  $a^{k+1}$  such that  $\langle a^i \rangle_{i=1}^{k+1} \in H$  are called **terminal histories**  $Z$ . Components of a history are called **actions**.

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## Definition (Extensive game with perfect information, ctd.)

- A **player function**  $P : H \setminus Z \rightarrow N$  that determines which player's turn it is to move after a given nonterminal history.
- For each player  $i \in N$ , a **utility function** (or **payoff function**)  $u_i : Z \rightarrow \mathbb{R}$  defined on the set of terminal histories.

The game is called **finite**, if  $H$  is finite. It has a **finite horizon**, if the length of histories is bounded from above.

**Assumption:** All ingredients of  $\Gamma$  are **common knowledge** amongst the players of the game.

**Terminology:** In the following, we will simply write **extensive games** instead of **extensive games with perfect information**.

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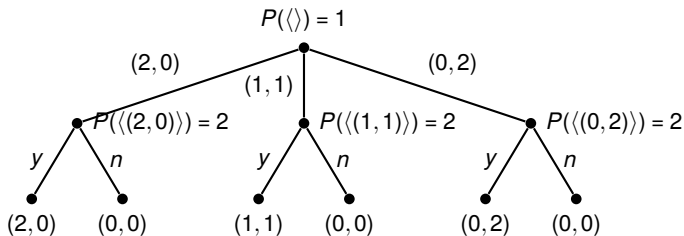
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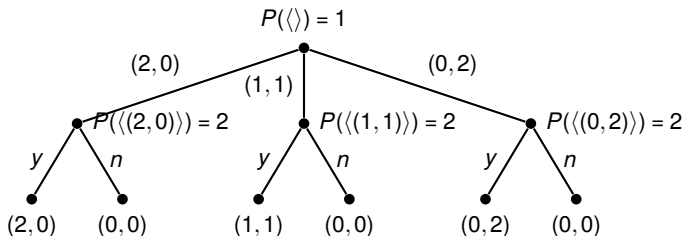
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## Example (Division game)

- Two identical objects should be **divided** among two players.
- **Player 1 proposes** an allocation.
- **Player 2 agrees or rejects**.
  - **On agreement:** Allocation as proposed.
  - **On rejection:** Nobody gets anything.



## Example (Division game, formally)



- $N = \{1, 2\}$
- $H = \{\langle \rangle, \langle (2,0) \rangle, \langle (1,1) \rangle, \langle (0,2) \rangle, \langle (2,0), y \rangle, \langle (2,0), n \rangle, \dots\}$
- $P(\langle \rangle) = 1, P(h) = 2$  for all  $h \in H \setminus Z$  with  $h \neq \langle \rangle$
- $u_1(\langle (2,0), y \rangle) = 2, u_2(\langle (2,0), y \rangle) = 0$ , etc.

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## Notation:

Let  $h = \langle a^1, \dots, a^k \rangle$  be a history, and  $a$  an action.

- Then  $(h, a)$  is the history  $\langle a^1, \dots, a^k, a \rangle$ .
- If  $h' = \langle b^1, \dots, b^\ell \rangle$ , then  $(h, h')$  is the history  $\langle a^1, \dots, a^k, b^1, \dots, b^\ell \rangle$ .
- The set of actions from which player  $P(h)$  can choose after a history  $h \in H \setminus Z$  is written as

$$A(h) = \{a \mid (h, a) \in H\}.$$

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## Definition (Strategy in an extensive game)

A **strategy** of a player  $i$  in an extensive game  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  is a function  $s_i$  that assigns to each nonterminal history  $h \in H \setminus Z$  with  $P(h) = i$  an action  $a \in A(h)$ . The set of strategies of player  $i$  is denoted as  $S_i$ .

**Remark:** Strategies require us to assign actions to histories  $h$ , even if it is clear that they will never be played (e. g., because  $h$  will never be reached because of some earlier action).

**Notation (for finite games):** A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.

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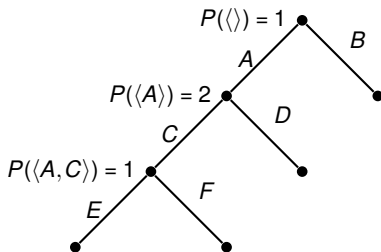
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## Example (Strategies in an extensive game)



- Strategies for player 1:  $AE$ ,  $AF$ ,  $BE$  and  $BF$
- Strategies for player 2:  $C$  and  $D$ .

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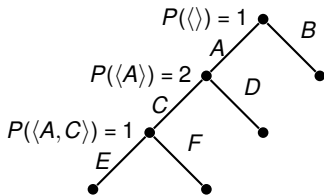
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## Definition (Outcome)

The **outcome**  $O(s)$  of a strategy profile  $s = (s_i)_{i \in N}$  is the (possibly infinite) terminal history  $h = \langle a^i \rangle_{i=1}^k$ , with  $k \in \mathbb{N} \cup \{\infty\}$ , such that for all  $\ell \in \mathbb{N}$  with  $0 \leq \ell < k$ ,

$$s_{P(\langle a^1, \dots, a^\ell \rangle)}(\langle a^1, \dots, a^\ell \rangle) = a^{\ell+1}.$$

## Example (Outcome)



$$O(AF, C) = \langle A, C, F \rangle$$

$$O(AE, D) = \langle A, D \rangle.$$

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## Definition (Nash equilibrium in an extensive game)

A **Nash equilibrium** in an extensive game  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  is a strategy profile  $s^*$  such that for every player  $i \in N$  and for all strategies  $s_i \in S_i$ ,

$$u_i(O(s_{-i}^*, s_i^*)) \geq u_i(O(s_{-i}^*, s_i)).$$

## Definition (Induced strategic game)

The strategic game  $G$  **induced** by an extensive game  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  is defined by  $G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$ , where

- $A'_i = S_i$  for all  $i \in N$ , and
- $u'_i(a) = u_i(O(a))$  for all  $i \in N$ .

## Proposition

The Nash equilibria of an extensive game  $\Gamma$  are exactly the Nash equilibria of the induced strategic game  $G$  of  $\Gamma$ . □

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## Remarks:

- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.

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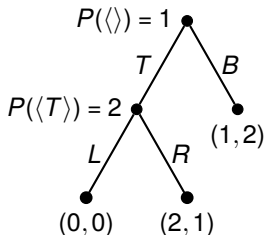
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## Example (Empty threat)

Extensive game:



Strategic form:

	L	R
T	0, 0	2, 1
B	1, 2	1, 2

Nash equilibria:  $(B, L)$  and  $(T, R)$ .

However,  $(B, L)$  is not realistic:

- Player 1 plays  $B$ , “fearing” response  $L$  to  $T$ .
- But player 2 would never play  $L$  in the extensive game.  
 $\rightsquigarrow (B, L)$  involves “empty threat”.

Strategies:

- Player 1:  $T$  and  $B$
- Player 2:  $L$  and  $R$

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**Idea:** Exclude empty threats.

**How?** Demand that a strategy profile is not only a Nash equilibrium in the strategic form, but also in every subgame.

## Definition (Subgame)

A **subgame** of an extensive game  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ , starting after history  $h$ , is the game  $\Gamma(h) = \langle N, H|_h, P|_h, (u_i|_h)_{i \in N} \rangle$ , where

- $H|_h = \{h' \mid (h, h') \in H\}$ ,
- $P|_h(h') = P(h, h')$  for all  $h' \in H|_h$ , and
- $u_i|_h(h') = u_i(h, h')$  for all  $h' \in H|_h$ .

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## Definition (Strategy in a subgame)

Let  $\Gamma$  be an extensive game and  $\Gamma(h)$  a subgame of  $\Gamma$  starting after some history  $h$ .

For each strategy  $s_i$  of  $\Gamma$ , let  $s_i|_h$  be the strategy induced by  $s_i$  for  $\Gamma(h)$ . Formally, for all  $h' \in H|_h$ ,

$$s_i|_h(h') = s_i(h, h').$$

The outcome function of  $\Gamma(h)$  is denoted by  $O_h$ .



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## Definition (Subgame-perfect equilibrium)

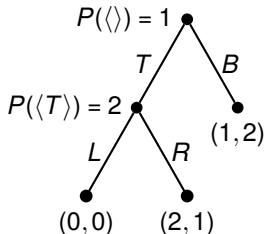
A strategy profile  $s^*$  in an extensive game  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  is a **subgame-perfect equilibrium** if and only if for every player  $i \in N$  and every nonterminal history  $h \in H \setminus Z$  with  $P(h) = i$ ,

$$u_i|_h(O_h(s^*_{-i}|_h, s_i^*|_h)) \geq u_i|_h(O_h(s^*_{-i}|_h, s_i))$$

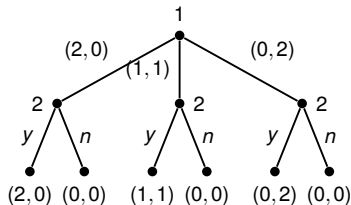
for every strategy  $s_i \in S_i$  in subgame  $\Gamma(h)$ .

Two Nash equilibria:

- $(T, R)$ : **subgame-perfect**, because:
  - In history  $h = \langle T \rangle$ : subgame-perfect.
  - In history  $h = \langle \rangle$ : player 1 obtains utility 1 when choosing  $B$  and utility of 2 when choosing  $T$ .
- $(B, L)$ : **not subgame-perfect**, since  $L$  does not maximize the utility of player 2 in history  $h = \langle T \rangle$ .



## Example (Subgame-perfect equilibria in division game)



### Equilibria in subgames:

- in  $\Gamma(\langle(2,0)\rangle)$ :  $y$  and  $n$
- in  $\Gamma(\langle(1,1)\rangle)$ : only  $y$
- in  $\Gamma(\langle(0,2)\rangle)$ : only  $y$
- in  $\Gamma(\langle\rangle)$ :  $((2,0), yyy)$  and  $((1,1), nyy)$

### Nash equilibria (red: empty threat):

- $((2,0), yyy)$ ,  $((2,0), yyn)$ ,  $((2,0), yny)$ ,  $((2,0), ynn)$ ,  $((2,0), nny)$ ,  $((2,0), nnn)$ ,
- $((1,1), nyy)$ ,  $((1,1), nyn)$ .
- $((0,2), nny)$

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# 4 One-Deviation Property



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- **Existence:**
  - Does every extensive game have a subgame-perfect equilibrium?
  - If not, which extensive games do have a subgame-perfect equilibrium?
- **Computation:**
  - If a subgame-perfect equilibrium exists, how to compute it?
  - How complex is that computation?

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Positive case (a subgame-perfect equilibrium exists):

- **Step 1:** Show that it suffices to consider **local** deviations from strategies (for finite-horizon games).
- **Step 2:** Show how to **systematically explore such local deviations** to find a subgame-perfect equilibrium (for finite games).

# Step 1: One-Deviation Property



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## Definition

Let  $\Gamma$  be a finite-horizon extensive game. Then  $\ell(\Gamma)$  denotes the length of the longest history of  $\Gamma$ .

# Step 1: One-Deviation Property



## Definition (One-deviation property)

A strategy profile  $s^*$  in an extensive game  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  satisfies the one-deviation property if and only if for every player  $i \in N$  and every nonterminal history  $h \in H \setminus Z$  with  $P(h) = i$ ,

$$u_i|_h(O_h(s_{-i}^*|_h, s_i^*|_h)) \geq u_i|_h(O_h(s_{-i}^*|_h, s_i))$$

for every strategy  $s_i \in S_i$  in subgame  $\Gamma(h)$  that differs from  $s_i^*|_h$  only in the action it prescribes after the initial history of  $\Gamma(h)$ .

**Note:** Without the highlighted parts, this is just the definition of subgame-perfect equilibria!

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# Step 1: One-Deviation Property



## Lemma

Let  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  be a **finite-horizon** extensive game. Then a strategy profile  $s^*$  is a subgame-perfect equilibrium of  $\Gamma$  if and only if it satisfies the one-deviation property.

## Proof

- $(\Rightarrow)$  Clear.
- $(\Leftarrow)$  By contradiction:

Suppose that  $s^*$  is not a subgame-perfect equilibrium.

Then there is a history  $h$  and a player  $i$  such that  $s_i$  is a profitable deviation for player  $i$  in subgame  $\Gamma(h)$ .

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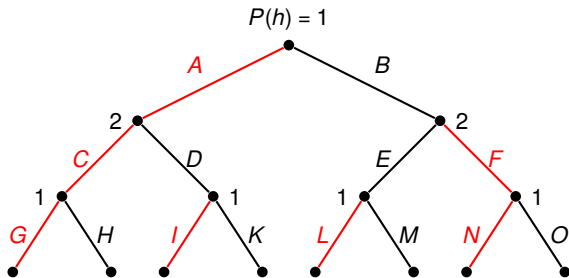
# Step 1: One-Deviation Property



## Proof (ctd.)

- ( $\Leftarrow$ ) ... WLOG, the number of histories  $h'$  with  $s_i(h') \neq s_i^*|_h(h')$  is at most  $\ell(\Gamma(h))$  and hence finite (finite horizon assumption!), since deviations not on resulting outcome path are irrelevant.

Illustration: strategies  $s_1^*|_h = AGILN$  and  $s_2^*|_h = CF$  red:



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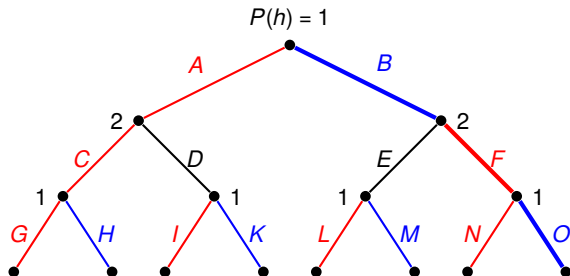
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# Step 1: One-Deviation Property



## Proof (ctd.)

- ( $\Leftarrow$ ) ... Illustration for WLOG assumption: Assume  $s_1 = BHKMO$  (blue) profitable deviation:



Then only  $B$  and  $O$  really matter.

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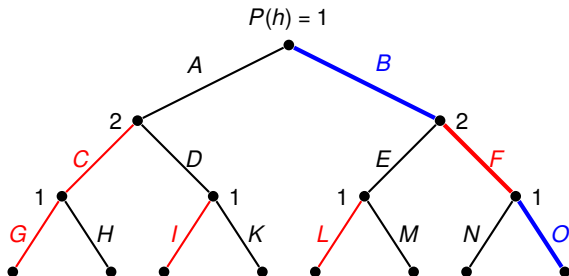
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# Step 1: One-Deviation Property



## Proof (ctd.)

- ( $\Leftarrow$ ) ... Illustration for WLOG assumption: And hence  $\tilde{s}_1 = BGILO$  (blue) also profitable deviation:



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# Step 1: One-Deviation Property



## Proof (ctd.)

### ■ ( $\Leftarrow$ ) ...

Choose profitable deviation  $s_i$  in  $\Gamma(h)$  with minimal number of deviation points (such  $s_i$  must exist).

Let  $h^*$  be the longest history in  $\Gamma(h)$  with  $s_i(h^*) \neq s_i^*|_h(h^*)$ , i.e., “deepest” deviation point for  $s_i$ .

Then in  $\Gamma(h, h^*)$ ,  $s_i|_{h^*}$  differs from  $s_i^*|_{(h, h^*)}$  only in the initial history.

Moreover,  $s_i|_{h^*}$  is a profitable deviation in  $\Gamma(h, h^*)$ , since  $h^*$  is the *longest* history in  $\Gamma(h)$  with  $s_i(h^*) \neq s_i^*|_h(h^*)$ .

So,  $\Gamma(h, h^*)$  is the desired subgame where a one-step deviation is sufficient to improve utility. □

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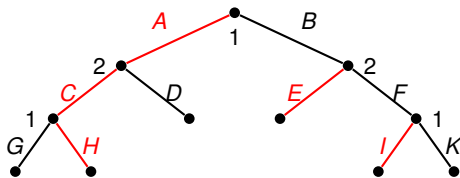
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# Step 1: One-Deviation Property

## Example



To show that  $(AHI, CE)$  is a subgame-perfect equilibrium, it suffices to check these deviating strategies:

### Player 1:

- $G$  in subgame  $\Gamma(\langle A, C \rangle)$
- $K$  in subgame  $\Gamma(\langle B, F \rangle)$
- $BHI$  in  $\Gamma$

### Player 2:

- $D$  in subgame  $\Gamma(\langle A \rangle)$
- $F$  in subgame  $\Gamma(\langle B \rangle)$

In particular, e.g., no need to check if strategy  $BGK$  of player 1 is profitable in  $\Gamma$ .

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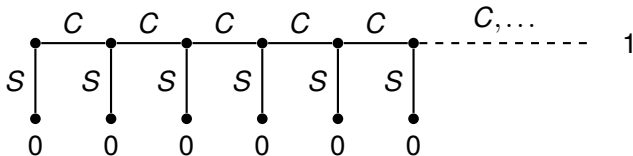
# Step 1: One-Deviation Property

## Remark on Infinite-Horizon Games



The corresponding proposition for infinite-horizon games **does not hold**.

Counterexample (one-player case):



Strategy  $s_i$  with  $s_i(h) = S$  for all  $h \in H \setminus Z$

- satisfies one deviation property, but
- is not a subgame-perfect equilibrium, since it is dominated by  $s_i^*$  with  $s_i^*(h) = C$  for all  $h \in H \setminus Z$ .

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# 5 Kuhn's Theorem



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### Theorem (Kuhn)

Every **finite** extensive game has a subgame-perfect equilibrium.

#### Proof idea:

- Proof is **constructive** and builds a subgame-perfect equilibrium bottom-up (aka **backward induction**).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.

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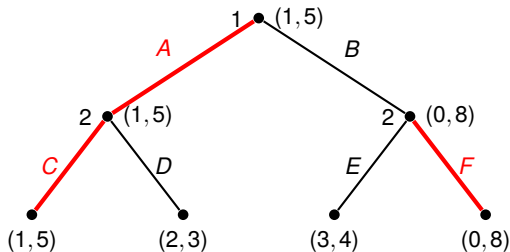
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## Step 2: Kuhn's Theorem



### Example



$$s_2(\langle A \rangle) = C$$

$$t_1(\langle A \rangle) = 1$$

$$t_2(\langle A \rangle) = 5$$

$$s_2(\langle B \rangle) = F$$

$$t_1(\langle B \rangle) = 0$$

$$t_2(\langle B \rangle) = 8$$

$$s_1(\langle \rangle) = A$$

$$t_1(\langle \rangle) = 1$$

$$t_2(\langle \rangle) = 5$$

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## Step 2: Kuhn's Theorem

A bit more formally:

### Proof

Let  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$  be a finite extensive game.

Construct a subgame-perfect equilibrium by induction on  $\ell(\Gamma(h))$  for all subgames  $\Gamma(h)$ . In parallel, construct functions  $t_i : H \rightarrow \mathbb{R}$  for all players  $i \in N$  s. t.  $t_i(h)$  is the payoff for player  $i$  in a subgame-perfect equilibrium in subgame  $\Gamma(h)$ .

**Base case:** If  $\ell(\Gamma(h)) = 0$ , then  $t_i(h) = u_i(h)$  for all  $i \in N$ .

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## Step 2: Kuhn's Theorem

### Proof (ctd.)

**Inductive case:** If  $t_j(h)$  already defined for all  $h \in H$  with  $\ell(\Gamma(h)) \leq k$ , consider  $h^* \in H$  with  $\ell(\Gamma(h^*)) = k + 1$  and  $P(h^*) = i$ .

For all  $a \in A(h^*)$ ,  $\ell(\Gamma(h^*, a)) \leq k$ , let

$$s_i(h^*) := \operatorname{argmax}_{a \in A(h^*)} t_i(h^*, a) \quad \text{and}$$

$$t_j(h^*) := t_j(h^*, s_i(h^*)) \quad \text{for all players } j \in N.$$

Inductively, we obtain a strategy profile  $s$  that satisfies the one-deviation property.

With the one-deviation property lemma it follows that  $s$  is a subgame-perfect equilibrium. □

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## Step 2: Kuhn's Theorem



- **In principle:** sample subgame-perfect equilibrium effectively computable using the technique from the above proof.
- **In practice:** often game trees not enumerated in advance, hence unavailable for backward induction.
- E.g., for branching factor  $b$  and depth  $m$ , procedure needs time  $O(b^m)$ .

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# Step 2: Kuhn's Theorem

## Remark on Infinite Games



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Corresponding proposition for infinite games **does not hold**.

Counterexamples (both for one-player case):

A) finite horizon, infinite branching factor:

Infinitely many actions  $a \in A = [0, 1)$  with payoffs  $u_1(\langle a \rangle) = a$  for all  $a \in A$ .

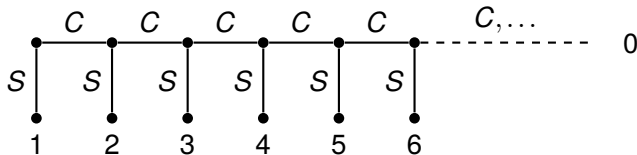
There exists no subgame-perfect equilibrium in this game.

# Step 2: Kuhn's Theorem

## Remark on Infinite Games



B) infinite horizon, finite branching factor:



$$u_1(CCC\dots) = 0 \text{ and } u_1(\underbrace{CC\dots C}_n S) = n + 1.$$

No subgame-perfect equilibrium.

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### Uniqueness:

Kuhn's theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.

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# Extended Example: Pirate Game



- 1 There are 5 *rational* pirates,  $A, B, C, D$  and  $E$ . They find 100 gold coins. They must decide how to distribute them.
- 2 The pirates have a strict order of *seniority*:  $A$  is senior to  $B$ , who is senior to  $C$ , who is senior to  $D$ , who is senior to  $E$ .
- 3 The pirate world's rules of distribution say that the most senior pirate first *proposes* a distribution of coins. The pirates, including the proposer, then *vote* on whether to accept this distribution (in order from most junior to senior). In case of a tie vote, the proposer has the casting vote. If the distribution is accepted, the coins are disbursed and the *game ends*. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to apply the method again.

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- 4 The pirates do not trust each other, and will neither make nor honor any promises between pirates apart from a proposed distribution plan that gives a whole number of gold coins to each pirate.
- 5 Pirates base their decisions on three factors. First of all, each pirate wants to *survive*. Second, everything being equal, each pirate wants to *maximize the number of gold coins* each receives. Third, each pirate would prefer to *throw another overboard*, if all other results would otherwise be equal.

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- Players  $N = \{A, B, C, D, E\}$ ;
- actions are:
  - proposals by a pirate:  $\langle A : x_A, B : x_B, C : x_C, D : x_D, E : x_E \rangle$ , with  $\sum_{i \in \{A, B, C, D, E\}} x_i = 100$ ;
  - votings:  $y$  for accepting,  $n$  for rejecting;
- histories are sequences of a proposal, followed by votings of the alive pirates;
- utilities:
  - for pirates who are alive: utilities are according to the accepted proposal plus  $x/100$ ,  $x$  being the number of dead pirates;
  - for dead pirates: -100.

**Remark:** Very large game tree!

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- 1 Assume only  $D$  and  $E$  are still alive.  $D$  can propose  $\langle A : 0, B : 0, C : 0, D : 100, E : 0 \rangle$ , because  $D$  has the casting vote!
- 2 Assume  $C$ ,  $D$ , and  $E$  are alive. For  $C$  it is enough to offer 1 coin to  $E$  to get his vote:  $\langle A : 0, B : 0, C : 99, D : 0, E : 1 \rangle$ .
- 3 Assume  $B$ ,  $C$ ,  $D$ , and  $E$  are alive.  $B$  offering  $D$  one coin is enough because of the casting vote:  
 $\langle A : 0, B : 99, C : 0, D : 1, E : 0 \rangle$ .
- 4 Assume  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  are alive.  $A$  offering  $C$  and  $E$  each one coin is enough:  $\langle A : 98, B : 0, C : 1, D : 0, E : 1 \rangle$   
(note that giving 1 to  $D$  instead to  $E$  does not help).

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# 6 Two Extensions

- Simultaneous Moves
- Chance

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## Definition

An **extensive game with simultaneous moves** is a tuple  $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ , where

- $N, H, P$  and  $(u_i)$  are defined as before, and
- $P : H \rightarrow 2^N$  assigns to each nonterminal history a **set** of players to move; for all  $h \in H \setminus Z$ , there exists a family  $(A_i(h))_{i \in P(h)}$  such that

$$A(h) = \{a \mid (h, a) \in H\} = \prod_{i \in P(h)} A_i(h).$$

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- **Intended meaning of simultaneous moves:** All players from  $P(h)$  move simultaneously.
- **Strategies:** Functions  $s_i : h \mapsto a_i$  with  $a_i \in A_i(h)$ .
- **Histories:** Sequences of vectors of actions.
- **Outcome:** Terminal history reached when tracing strategy profile.
- **Payoffs:** Utilities at outcome history.

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### Remark:

- The **one-deviation property still holds** for extensive game with perfect information and simultaneous moves.
- **Kuhn's theorem does not hold** for extensive game with simultaneous moves.

**Example:** MATCHING PENNIES can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

		player 2	
		<i>H</i>	<i>T</i>
player 1	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

⇒ Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this lecture.

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# Simultaneous Moves

## Example: Three-Person Cake Splitting Game



### Setting:

- Three players have to split a cake fairly.
- Player 1 suggest split: shares  $x_1, x_2, x_3 \in [0, 1]$  s.t.  
 $x_1 + x_2 + x_3 = 1$ .
- Then players 2 and 3 **simultaneously** and **independently** decide whether to accept (“y”) or reject (“n”) the suggested splitting.
- If both accept, each player  $i$  gets his allotted share (utility  $x_i$ ). Otherwise, no player gets anything (utility 0).

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# Simultaneous Moves

## Example: Three-Person Cake Splitting Game



Formally:

$$N = \{1, 2, 3\}$$

$$X = \{(x_1, x_2, x_3) \in [0, 1]^3 \mid x_1 + x_2 + x_3 = 1\}$$

$$H = \{\langle \rangle\} \cup \{\langle x \rangle \mid x \in X\} \cup \{\langle x, z \rangle \mid x \in X, z \in \{y, n\} \times \{y, n\}\}$$

$$P(\langle \rangle) = \{1\}$$

$$P(\langle x \rangle) = \{2, 3\} \text{ for all } x \in X$$

$$u_i(\langle x, z \rangle) = \begin{cases} 0 & \text{if } z \in \{(y, n), (n, y), (n, n)\} \\ x_i & \text{if } z = (y, y). \end{cases} \text{ for all } i \in N$$

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### Subgame-perfect equilibria:

- **Subgames after legal split**  $(x_1, x_2, x_3)$  by player 1:
  - NE  $(y, y)$  (both accept)
  - NE  $(n, n)$  (neither accepts)
  - If  $x_2 = 0$ , NE  $(n, y)$  (only player 3 accepts)
  - If  $x_3 = 0$ , NE  $(y, n)$  (only player 2 accepts)

### Subgame-perfect equilibria (ctd.):

#### ■ Entire game:

Let  $s_2$  and  $s_3$  be any two strategies of players 2 and 3 such that for all splits  $x \in X$  the profile  $(s_2(\langle x \rangle), s_3(\langle x \rangle))$  is one of the NEs from above.

Let  $X_y = \{x \in X \mid s_2(\langle x \rangle) = s_3(\langle x \rangle) = y\}$  be the set of splits accepted under  $s_2$  and  $s_3$ . Distinguish three cases:

- $X_y = \emptyset$  or  $x_1 = 0$  for all  $x \in X_y$ . Then  $(s_1, s_2, s_3)$  is a subgame-perfect equilibrium for any possible  $s_1$ .
- $X_y \neq \emptyset$  and there are splits  $x_{\max} = (x_1, x_2, x_3) \in X_y$  that maximize  $x_1 > 0$ . Then  $(s_1, s_2, s_3)$  is a subgame-perfect equilibrium if and only if  $s_1(\langle \cdot \rangle)$  is such a split  $x_{\max}$ .
- $X_y \neq \emptyset$  and there are no splits  $(x_1, x_2, x_3) \in X_y$  that maximize  $x_1$ . Then there is no subgame-perfect equilibrium, in which player 2 follows strategy  $s_2$  and player 3 follows strategy  $s_3$ .

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## Definition

An **extensive game with chance moves** is a tuple

$\Gamma = \langle N, H, P, f_c, (u_i)_{i \in N} \rangle$ , where

- $N, A, H$  and  $u_i$  are defined as before,
- the player function  $P : H \setminus Z \rightarrow N \cup \{c\}$  can also take the value **c** for a **chance node**, and
- for each  $h \in H \setminus Z$  with  $P(h) = c$ , the function  $f_c(\cdot|h)$  is a probability distribution on  $A(h)$  such that the probability distributions for all  $h \in H$  are independent of each other.

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- **Intended meaning of chance moves:** In chance node, an applicable action is chosen randomly with probability according to  $f_c$ .
- **Strategies:** Defined as before.
- **Outcome:** For a given strategy profile, the outcome is a probability distribution on the set of terminal histories.
- **Payoffs:** For player  $i$ ,  $U_i$  is the expected payoff (with weights according to outcome probabilities).

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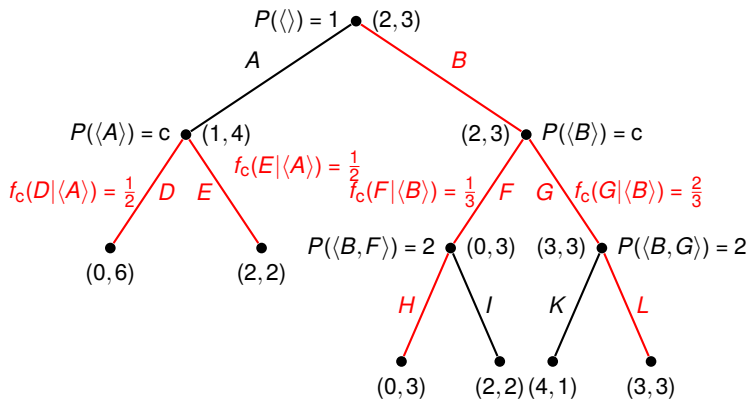
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### Remark:

The one-deviation property and Kuhn's theorem still hold in the presence of chance moves. When proving Kuhn's theorem, **expected** utilities have to be used.

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**Summary**



- For **finite-horizon extensive games**, it suffices to consider **local deviations** when looking for better strategies.
- For infinite-horizon games, this is not true in general.
- Every **finite extensive game** has a **subgame-perfect equilibrium**.
- This does not generally hold for infinite games, no matter if game is infinite due to infinite branching factor or infinitely long histories (or both).
  
- With **chance moves**, one deviation property and Kuhn's theorem still hold.
- With **simultaneous moves**, Kuhn's theorem no longer holds.

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