

# Game Theory

## 4. Algorithms

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# Motivation

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Zero-Sum Games

General Finite Two-Player Games

Summary



- **We know:** In finite strategic games, mixed-strategy Nash equilibria are guaranteed to **exist**.
- **We don't know:** How to systematically **find them**?
- **Challenge:** There are **infinitely many** mixed strategy profiles to consider. How to do this in finite time?

## This chapter:

- Computation of mixed-strategy Nash equilibria for **finite zero-sum games**.
- Computation of mixed-strategy Nash equilibria for **general finite two player games**.

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# Linear Programming

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## Digression:

We briefly discuss **linear programming** because we will use this technique to find Nash equilibria.

## Goal of linear programming:

**Solving a system of linear inequalities** over  $n$  real-valued variables while **optimizing** some **linear objective function**.

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## Example

Production of two sorts of items with time requirements and profit per item. Objective: Maximize profit.

	Cutting	Assembly	Postproc.	Profit per item
( $x$ ) sort 1	25	60	68	30
( $y$ ) sort 2	75	60	34	40
per day	$\leq 450$	$\leq 480$	$\leq 476$	maximize!

**Goal:** Find numbers of pieces  $x$  of sort 1 and  $y$  of sort 2 to be produced per day such that the resource constraints are met and the objective function is maximized.

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## Example (ctd., formalization)

$$x \geq 0, y \geq 0 \quad (1)$$

$$25x + 75y \leq 450 \quad (\text{or } y \leq 6 - 1/3 x) \quad (2)$$

$$60x + 60y \leq 480 \quad (\text{or } y \leq 8 - x) \quad (3)$$

$$68x + 34y \leq 476 \quad (\text{or } y \leq 14 - 2x) \quad (4)$$

$$\text{maximize } z = 30x + 40y \quad (5)$$

- Inequalities (1)–(4): **Admissible solutions**  
(They form a **convex set** in  $\mathbb{R}^2$ .)
- Line (5): **Objective function**

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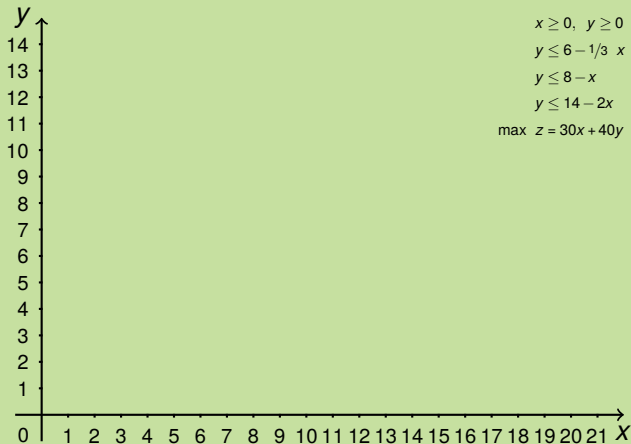
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## Example (ctd., visualization)



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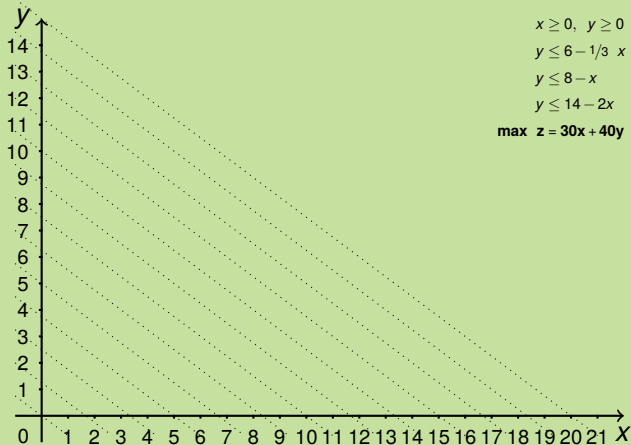
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## Example (ctd., visualization)



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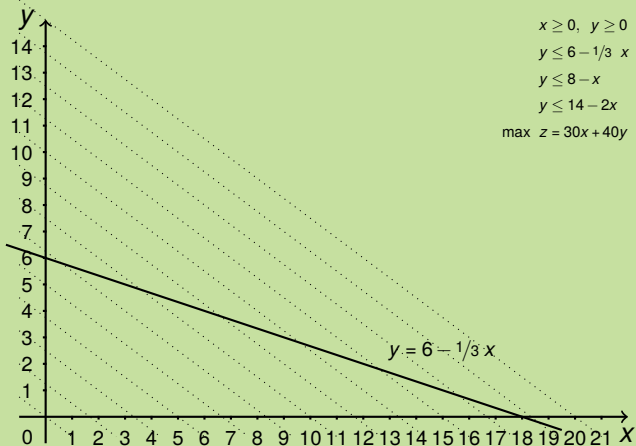
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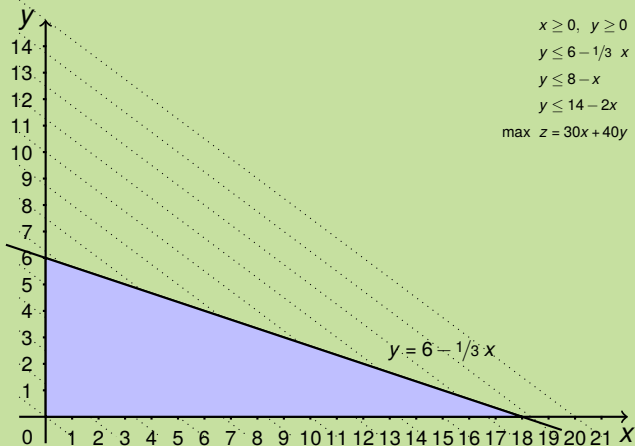
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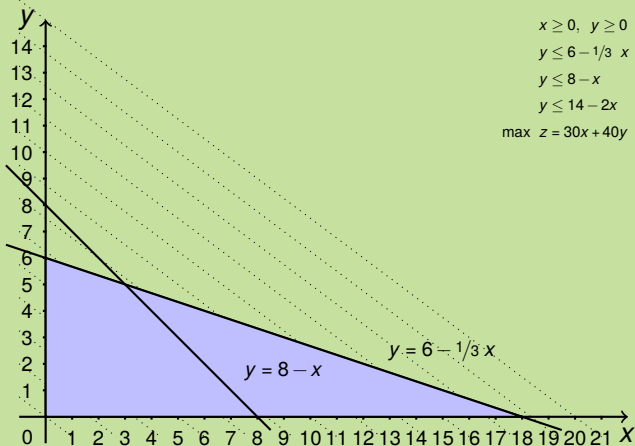
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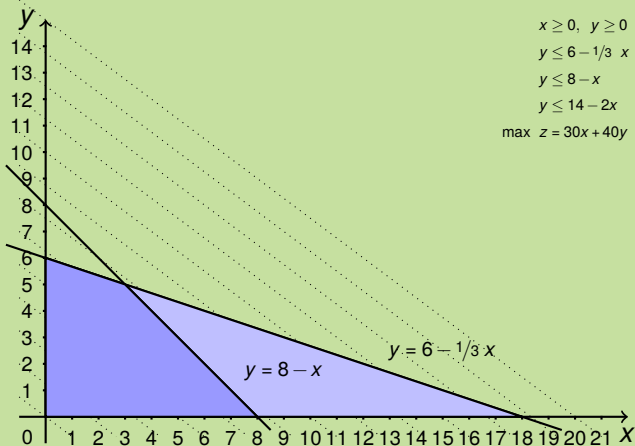
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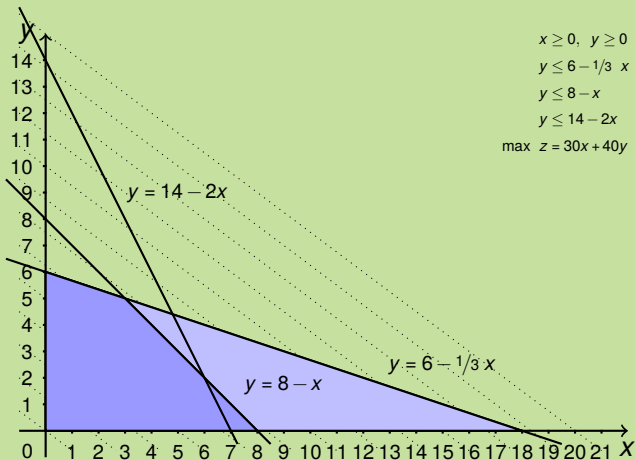
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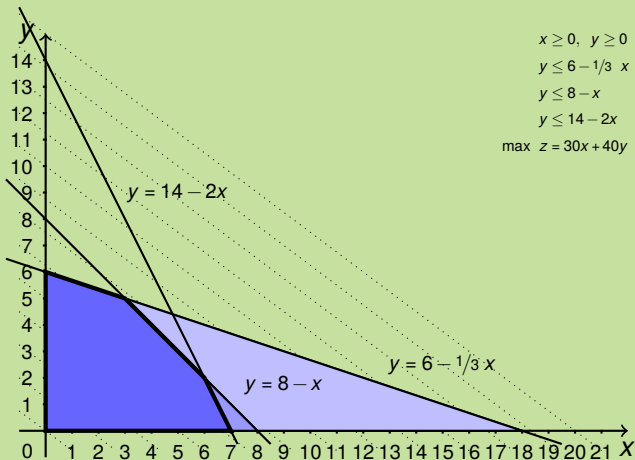
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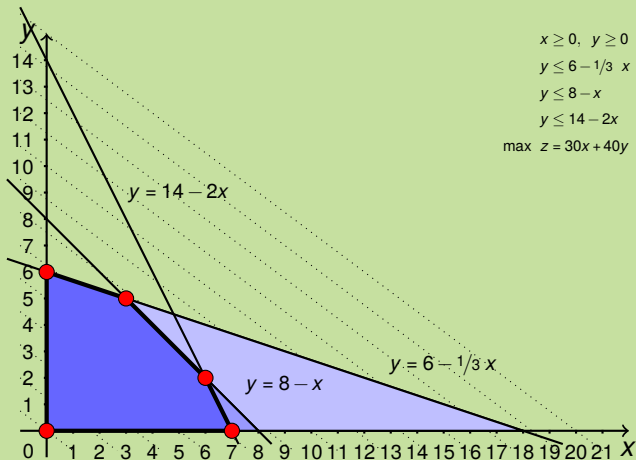
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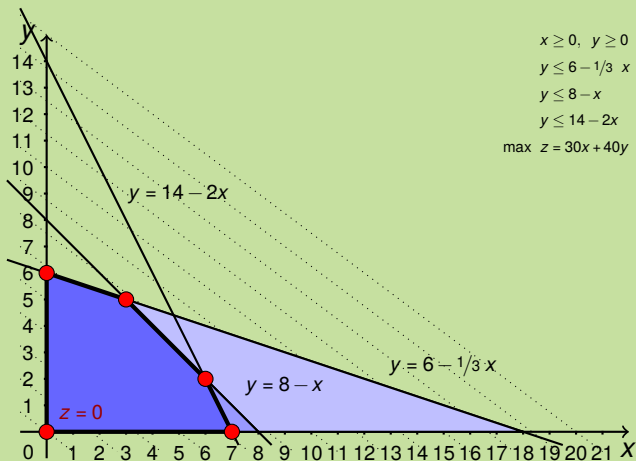
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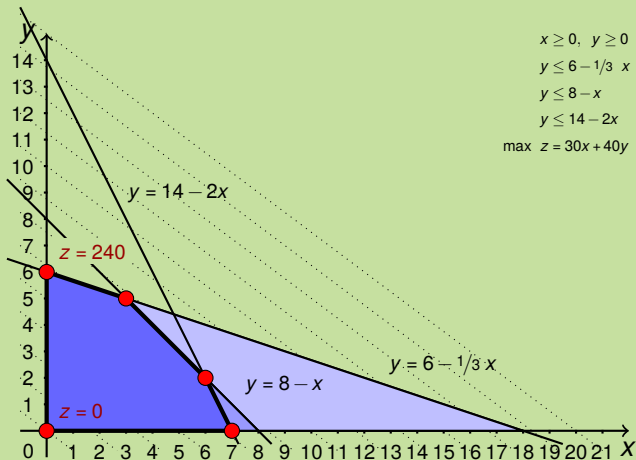
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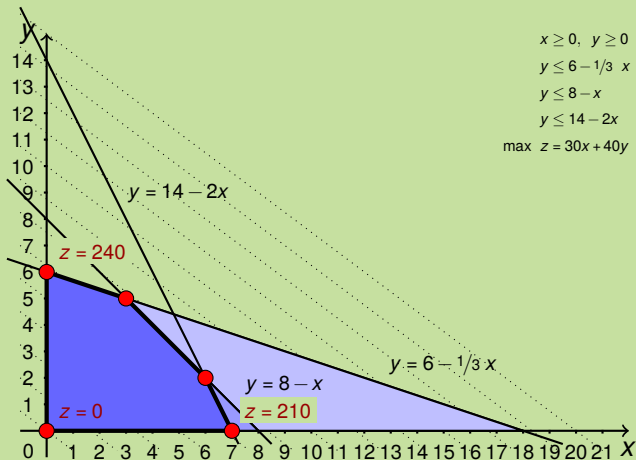
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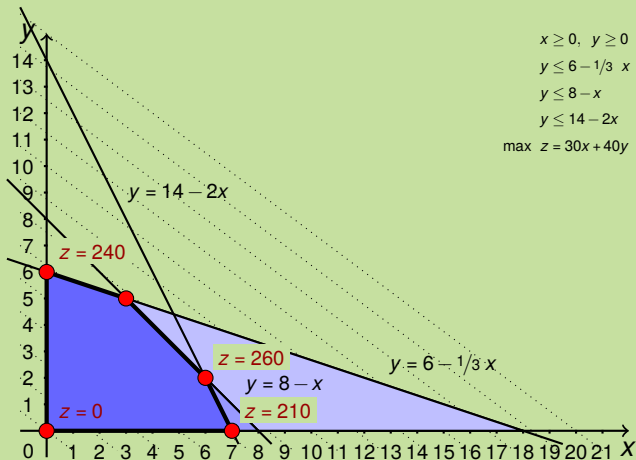
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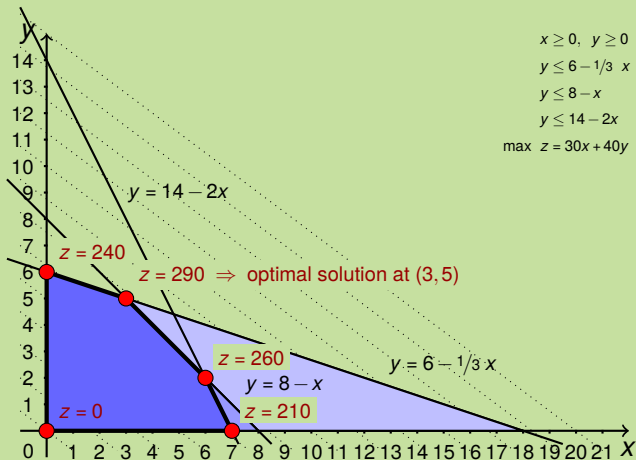
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## Definition (Linear program)

A **linear program** (LP) in standard form consists of

- $n$  real-valued variables  $x_i$ ;  $n$  coefficients  $b_i$ ;
- $m$  constants  $c_j$ ;  $n \cdot m$  coefficients  $a_{ij}$ ;
- $m$  constraints of the form

$$c_j \leq \sum_{i=1}^n a_{ij} x_i,$$

- and an objective function to be minimized ( $x_i \geq 0$ )

$$\sum_{i=1}^n b_i x_i.$$

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## Solution of an LP:

assignment of values to the  $x_i$  **satisfying the constraints** and **minimizing the objective function**.

## Remarks:

- **Maximization instead of minimization:** easy, just change the signs of all the  $b_i$ 's,  $i = 1, \dots, n$ .
- **Equalities** instead of inequalities:  $x + y \leq c$  if and only if there is a  $z \geq 0$  such that  $x + y + z = c$  ( $z$  is called a **slack variable**).

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## Solution algorithms:

- Usually, one uses the **simplex algorithm** (which is worst-case exponential!).
- There are also polynomial-time algorithms such as interior-point or ellipsoid algorithms.

## Tools and libraries:

- **lp\_solve**
- CLP
- GLPK
- CPLEX
- gurobi

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# Zero-Sum Games

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# Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games



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We start with **finite zero-sum games** for two reasons:

- They are **easier to solve** than general finite two-player games.
- Understanding how to solve finite zero-sum games **facilitates understanding** how to solve general finite two-player games.

# Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games



In the following, we will **exploit the zero-sum property** of a game  $G$  when searching for mixed-strategy Nash equilibria. For that, we need the following result.

## Proposition

Let  $G$  be a finite zero-sum game. Then the mixed extension of  $G$  is also a zero-sum game.

## Proof.

Homework. □

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# Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games



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Let  $G$  be a finite zero-sum game with mixed extension  $G'$ .

Then we know the following:

- 1 Previous proposition implies:  $G'$  is also a zero-sum game.
- 2 Nash's theorem implies:  $G'$  has a Nash equilibrium.
- 3 Maximimizer theorem + (1) + (2) implies: Nash equilibria and pairs of maximinimizers in  $G'$  are the same.

# Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games



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Consequence:

When looking for mixed-strategy Nash equilibria in  $G$ , it is sufficient to look for pairs of maximinimizers in  $G'$ .

Method: Linear Programming



## Approach:

- Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a finite zero-sum game:
  - $N = \{1, 2\}$ .
  - $A_1$  and  $A_2$  are finite.
  - $U_1(\alpha, \beta) = -U_2(\alpha, \beta)$  for all  $\alpha \in \Delta(A_1), \beta \in \Delta(A_2)$ .
- Player 1 looks for a maximinimizer mixed strategy  $\alpha$ .
- For each possible  $\alpha$  of player 1:
  - Determine expected utility against best response of pl. 2. (Only need to consider **finitely many pure** candidates for best responses because of Support Lemma).
  - Maximize expected utility over all possible  $\alpha$ .

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- **Result:** maximinimizer  $\alpha$  for player 1 in  $G'$   
(= Nash equilibrium strategy for player 1)
- **Analogously:** obtain maximinimizer  $\beta$  for player 2 in  $G'$   
(= Nash equilibrium strategy for player 2)
- **With maximinimizer theorem:** we can combine  $\alpha$  and  $\beta$  into a **Nash equilibrium**.

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“For each possible  $\alpha$  of player 1, determine expected utility against best response of player 2, and maximize.”

translates to the following LP:

$$\begin{aligned} \alpha(a) &\geq 0 && \text{for all } a \in A_1 \\ \sum_{a \in A_1} \alpha(a) &= 1 \\ U_1(\alpha, b) = \sum_{a \in A_1} \alpha(a) \cdot u_1(a, b) &\geq u && \text{for all } b \in A_2 \end{aligned}$$

Maximize  $u$ .

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## Example (Matching pennies)

	<i>H</i>	<i>T</i>
<i>H</i>	1, -1	-1, 1
<i>T</i>	-1, 1	1, -1

Linear program for player 1:

Maximize  $u$  subject to the constraints

$$\alpha(H) \geq 0, \alpha(T) \geq 0, \alpha(H) + \alpha(T) = 1,$$

$$\alpha(H) \cdot u_1(H, H) + \alpha(T) \cdot u_1(T, H) = \alpha(H) - \alpha(T) \geq u,$$

$$\alpha(H) \cdot u_1(H, T) + \alpha(T) \cdot u_1(T, T) = -\alpha(H) + \alpha(T) \geq u.$$

**Solution:**  $\alpha(H) = \alpha(T) = 1/2, u = 0.$

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- **Remark:** There is an alternative encoding based on the observation that in zero-sum games that have a Nash equilibrium, maximinimization and minimaximization yield the same result.
- **Idea:** Formulate linear program with inequalities

$$U_1(a, \beta) \leq u \quad \text{for all } a \in A_1$$

and minimize  $u$ . Analogously for  $\beta$ .

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# General Finite Two-Player Games

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- For general finite two-player games, the **LP approach does not work**.
- Instead, use instances of the **linear complementarity problem (LCP)**:
  - Linear (in-)equalities as with LPs.
  - Additional constraints of the form  $x_i \cdot y_i = 0$  (or equivalently  $x_i = 0 \vee y_i = 0$ ) for variables  $X = \{x_1, \dots, x_k\}$  and  $Y = \{y_1, \dots, y_k\}$ , and  $i \in \{1, \dots, k\}$ .
  - **no** objective function.
- With LCPs, we can compute Nash equilibria for **arbitrary** finite two-player games.

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# General Finite Two-Player Games



Let  $A_1$  and  $A_2$  be finite and let  $(\alpha, \beta)$  be a Nash equilibrium with payoff profile  $(u, v)$ . Then consider this LCP encoding:

$$u - U_1(a, \beta) \geq 0 \quad \text{for all } a \in A_1 \quad (6)$$

$$v - U_2(\alpha, b) \geq 0 \quad \text{for all } b \in A_2 \quad (7)$$

$$\alpha(a) \cdot (u - U_1(a, \beta)) = 0 \quad \text{for all } a \in A_1 \quad (8)$$

$$\beta(b) \cdot (v - U_2(\alpha, b)) = 0 \quad \text{for all } b \in A_2 \quad (9)$$

$$\alpha(a) \geq 0 \quad \text{for all } a \in A_1 \quad (10)$$

$$\sum_{a \in A_1} \alpha(a) = 1 \quad (11)$$

$$\beta(b) \geq 0 \quad \text{for all } b \in A_2 \quad (12)$$

$$\sum_{b \in A_2} \beta(b) = 1 \quad (13)$$

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## Remarks about the encoding:

- In (8) and (9): for instance,

$$\alpha(a) \cdot (u - U_1(a, \beta)) = 0$$

if and only if

$$\alpha(a) = 0 \quad \text{or} \quad u - U_1(a, \beta) = 0.$$

This holds in every Nash equilibrium, because:

- if  $a \notin \text{supp}(\alpha)$ , then  $\alpha(a) = 0$ , and
  - if  $a \in \text{supp}(\alpha)$ , then  $a \in B_1(\beta)$ , thus  $U_1(a, \beta) = u$ .
- With additional variables, the above LCP formulation can be transformed into LCP normal form.

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## Theorem

*A mixed strategy profile  $(\alpha, \beta)$  with payoff profile  $(u, v)$  is a Nash equilibrium if and only if it is a solution to the LCP encoding over  $(\alpha, \beta)$  and  $(u, v)$ .*

## Proof.

- **Nash equilibria are solutions to the LCP: Obvious** because of the support lemma.
- **Solutions to the LCP are Nash equilibria:** Let  $(\alpha, \beta, u, v)$  be a solution to the LCP. Because of (10)–(13),  $\alpha$  and  $\beta$  are mixed strategies.

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## Proof (ctd.)

- Solutions to the LCP are Nash equilibria (ctd.): Because of (6),  $u$  is at least the maximal payoff over all possible pure responses, and because of (8),  $u$  is exactly the maximal payoff.

If  $\alpha(a) > 0$ , then, because of (8), the payoff for player 1 against  $\beta$  is  $u$ .

The linearity of the expected utility implies that  $\alpha$  is a best response to  $\beta$ .

Analogously, we can show that  $\beta$  is a best response to  $\alpha$  and hence  $(\alpha, \beta)$  is a Nash equilibrium with payoff profile  $(u, v)$ .



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## Naïve algorithm:

Enumerate all  $(2^n - 1) \cdot (2^m - 1)$  possible pairs of support sets.

For each such pair  $(\text{supp}(\alpha), \text{supp}(\beta))$ :

- Convert the LCP into an LP:
  - Linear (in-)equalities are preserved.
  - Constraints of the form  $\alpha(a) \cdot (u - U_1(a, \beta)) = 0$  are replaced by a new linear equality:
    - $u - U_1(a, \beta) = 0$ , if  $a \in \text{supp}(\alpha)$ , and
    - $\alpha(a) = 0$ , otherwise,
  - Analogously for  $\beta(b) \cdot (v - U_2(\alpha, b)) = 0$ .
  - Objective function: maximize constant zero function.
- Apply solution algorithm for LPs to the transformed program.

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- Runtime of the naïve algorithm:  $O(p(n+m) \cdot 2^{n+m})$ , where  $p$  is some polynomial.
- Better in practice: Lemke-Howson algorithm.
- Complexity:
  - unknown whether  $\text{LCP SOLVE} \in \mathbf{P}$ .
  - $\text{LCP SOLVE} \in \mathbf{NP}$  is clear (naïve algorithm can be seen as a nondeterministic polynomial-time algorithm).

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**Summary**



- Computation of mixed-strategy Nash equilibria for **finite zero-sum games** using **linear programs**.  
~> polynomial-time computation
- Computation of mixed-strategy Nash equilibria for **general finite two player games** using **linear complementarity problem**.  
~> computation in **NP**.

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