Game Theory

3. Mixed Strategies

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1 Mixed Strategies

- Definitions
- Support Lemma

Mixed Strategies

Observation: Not every strategic game has a pure-strategy Nash equilibrium (e. g. matching pennies).

Question:

- Can we do anything about that?
- Which strategy to play then?

Idea: Consider randomized strategies.

Notation

Let X be a set.

Then $\Delta(X)$ denotes the set of probability distributions over X.

That is, each $p \in \Delta(X)$ is a mapping $p : X \to [0, 1]$ with

$$\sum_{x \in X} p(x) = 1.$$

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A mixed strategy is a strategy where a player is allowed to randomize his action (throw a dice mentally and then act according to what he has decided to do for each outcome).

Definition (Mixed strategy)

Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game.

A mixed strategy of player i in G is a probability distribution $\alpha_i \in \Delta(A_i)$ over player i's actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing a_i.

Terminology: When we talk about strategies in A_i specifically, to distinguish them from mixed strategies, we sometimes also call them pure strategies.

Mixed Strategies

Definition (Mixed strategy profile)

A profile $\alpha = (\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ of mixed strategies induces a probability distribution p_α over $A = \prod_{i \in N} A_i$ as follows:

$$p_\alpha(a) = \prod_{i \in N} \alpha_i(a_i).$$

For $A' \subseteq A$, we define

$$p_\alpha(A') = \sum_{a \in A'} p_\alpha(a) = \sum_{a \in A'} \prod_{i \in N} \alpha_i(a_i).$$

Notation

Since each pure strategy $a_i \in A_i$ is equivalent to its induced mixed strategy $\hat{a}_i$

$$\hat{a}_i(a_i') = \begin{cases} 1 & \text{if } a_i' = a_i \\ 0 & \text{otherwise,} \end{cases}$$

we sometimes abuse notation and write a_i instead of $\hat{a}_i$.

Example (Mixed strategies for matching pennies)

	H	T
H	$1, -1$	$-1, \ \ 1$
T	$-1, \ \ 1$	$1, -1$

$$\alpha = (\alpha_1, \alpha_2), \ \ \alpha_1(H) = {}^2\!/_3, \ \ \alpha_1(T) = {}^1\!/_3, \ \ \alpha_2(H) = {}^1\!/_3, \ \ \alpha_2(T) = {}^2\!/_3.$$

This induces a probability distribution over $\{H, T\} \times \{H, T\}$:

$$
\begin{aligned}
p_\alpha(H,H) &= \alpha_1(H) \cdot \alpha_2(H) = {}^2\!/_9, & u_1(H,H) &= +1, \\
p_\alpha(H,T) &= \alpha_1(H) \cdot \alpha_2(T) = {}^4\!/_9, & u_1(H,T) &= -1, \\
p_\alpha(T,H) &= \alpha_1(T) \cdot \alpha_2(H) = {}^1\!/_9, & u_1(T,H) &= -1, \\
p_\alpha(T,T) &= \alpha_1(T) \cdot \alpha_2(T) = {}^2\!/_9, & u_1(T,T) &= +1.
\end{aligned}
$$

Expected Utility

Definition (Expected utility)

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile.

The expected utility of α for player i is

$$U_i(\alpha) = U_i\big((\alpha_j)_{j\in N}\big) := \sum_{a \in A} p_\alpha(a)\, u_i(a) = \sum_{a \in A} \Big(\prod_{j \in N} \alpha_j(a_j)\Big) u_i(a).$$

Example (Mixed strategies for matching pennies (ctd.))

The expected utilities for player 1 and player 2 are

$$U_1(\alpha_1, \alpha_2) = -{}^1\!/_9 \qquad \text{and} \qquad U_2(\alpha_1, \alpha_2) = +{}^1\!/_9.$$

Expected Utility

Remark: The expected utility functions U_i are linear in all mixed strategies.

Proposition

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile, $\beta_i, \gamma_i \in \Delta(A_i)$ mixed strategies, and $\lambda \in [0,1]$. Then

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1-\lambda)U_i(\alpha_{-i}, \gamma_i).$$

Moreover,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot U_i(\alpha_{-i}, a_i)$$

Proof.

Homework. □

Definition (Mixed extension)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

The mixed extension of G is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where

- $\Delta(A_i)$ is the set of probability distributions over A_i and
- $U_i : \prod_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns to each mixed strategy profile α the expected utility for player i according to the induced probability distribution p_α.

Definition (Nash equilibrium in mixed strategies)

Let G be a strategic game.

A Nash equilibrium in mixed strategies (or mixed-strategy Nash equilibrium) of G is a Nash equilibrium in the mixed extension of G.

Support

Intuition:

- It does not make sense to assign positive probability to a pure strategy that is not a best response to what the other players do.
- Claim: A profile of mixed strategies α is a Nash equilibrium if and only if everyone only plays best pure responses to what the others play.

Definition (Support)

Let α_i be a mixed strategy.

The support of α_i is the set

$$supp(\alpha_i) = \{a_i \in A_i \,|\, \alpha_i(a_i) > 0\}$$

of actions played with nonzero probability.

Support Lemma

Lemma (Support lemma)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game.

Then $\alpha^* \in \prod_{i \in N} \Delta(A_i)$ is a mixed-strategy Nash equilibrium in G if and only if for every player $i \in N$, every pure strategy in the support of α_i^* is a best response to α_{-i}^*.

For a single player–given all other players stick to their mixed strategies–it does not make a difference whether he plays the mixed strategy or whether he plays any single pure strategy from the support of the mixed strategy.

Example (Support lemma)

Matching pennies, strategy profile $\alpha = (\alpha_1, \alpha_2)$ with

$$\alpha_1(H) = {}^2\!/_3, \quad \alpha_1(T) = {}^1\!/_3, \quad \alpha_2(H) = {}^1\!/_3, \text{ and } \quad \alpha_2(T) = {}^2\!/_3.$$

For α to be a Nash equilibrium, both actions in $supp(\alpha_2) = \{H, T\}$ have to be best responses to α_1. Are they?

$$\begin{aligned} U_2(\alpha_1, H) &= \alpha_1(H) \cdot u_2(H,H) + \alpha_1(T) \cdot u_2(T,H) \\ &= {}^2\!/_3 \cdot (-1) + {}^1\!/_3 \cdot (+1) = -{}^1\!/_3, \\ U_2(\alpha_1, T) &= \alpha_1(H) \cdot u_2(H,T) + \alpha_1(T) \cdot u_2(T,T) \\ &= {}^2\!/_3 \cdot (+1) + {}^1\!/_3 \cdot (-1) = {}^1\!/_3. \end{aligned}$$

$\Rightarrow$ $H \in supp(\alpha_2)$, but $H \notin B_2(\alpha_1)$.

$\overset{\text{Support lemma}}{\Rightarrow}$ α can not be a Nash equilibrium.

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α_{-i}^*. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

This makes the modified α_i^* a better response than the original α_i^*, i. e., the original α_i^* was not a best response, which contradicts the assumption that α^* is a Nash equilibrium.

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α_i' such that $U_i(\alpha_{-i}^*, \alpha_i') > U_i(\alpha_{-i}^*, \alpha_i^*)$.

Because U_i is linear, there must be a pure strategy $a_i' \in supp(\alpha_i')$ that has higher utility than some pure strategy $a_i'' \in supp(\alpha_i^*)$.

Therefore, $supp(\alpha_i^*)$ does not only contain best responses to α_{-i}^*. $\square$

Example (Mixed-strategy Nash equilibria in BoS)

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

We already know: (B, B) and (S, S) are pure Nash equilibria.

Possible supports (excluding "pure-vs-pure" strategies) are:

$$\{B\} \text{ vs. } \{B, S\}, \quad \{S\} \text{ vs. } \{B, S\}, \quad \{B, S\} \text{ vs. } \{B\},$$
$$\{B, S\} \text{ vs. } \{S\} \quad \text{and} \quad \{B, S\} \text{ vs. } \{B, S\}$$

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibria of "pure-vs-strictly-mixed" type.

Example (Mixed-strategy Nash equilibria in BoS (ctd.))

Consequence: Only need to search for additional Nash equilibria with support sets $\{B, S\}$ vs. $\{B, S\}$.
Assume that (α_1^*, α_2^*) is a Nash equilibrium with $0 < \alpha_1^*(B) < 1$ and $0 < \alpha_2^*(B) < 1$. Then

$$
\begin{aligned}
& & U_1(B, \alpha_2^*) &= U_1(S, \alpha_2^*) \\
&\Rightarrow & 2 \cdot \alpha_2^*(B) + 0 \cdot \alpha_2^*(S) &= 0 \cdot \alpha_2^*(B) + 1 \cdot \alpha_2^*(S) \\
&\Rightarrow & 2 \cdot \alpha_2^*(B) &= 1 - \alpha_2^*(B) \\
&\Rightarrow & 3 \cdot \alpha_2^*(B) &= 1 \\
&\Rightarrow & \alpha_2^*(B) &= 1/3 \quad (\text{and } \alpha_2^*(S) = 2/3)
\end{aligned}
$$

Similarly, we get $\alpha_1^*(B) = 2/3$ and $\alpha_1^*(S) = 1/3$.
The payoff profile of this equilibrium is $(2/3, 2/3)$.

Support Lemma

Remark

Let $G = \langle\{1,2\},(A_i),(u_i)\rangle$ with $A_1 = \{T,B\}$ and $A_2 = \{L,R\}$ be a two-player game with two actions each, and (T,α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T,L) and (T,R) is also a Nash equilibrium of G.

Reason: Both L and R are best responses to T. Assume that T was neither a best response to L nor to R. Then B would be a better response than T both to L and to R.

With the linearity of U_1, B would also be a better response to α_2^* than T is. Contradiction.

Example

Consider the Nash equilibrium $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\alpha_1^*(T) = 1, \quad \alpha_1^*(B) = 0, \quad \alpha_2^*(L) = 1/10, \quad \alpha_2^*(R) = 9/10$$

in the following game:

	L	R
T	1, 1	1, 1
B	2, 2	−5, −5

Here, (T, R) is also a Nash equilibrium.

2 Nash's Theorem

- Definitions
- Kakutani's Fixpoint Theorem
- Proof of Nash's Theorem

Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

In the previous chapter, we discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?

Theorem (Nash's theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Proof sketch.

Consider the set-valued function of best responses $B : \mathbb{R}^{\sum_i |A_i|} \to 2^{\mathbb{R}^{\sum_i |A_i|}}$ with

$$B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}).$$

A mixed strategy profile α is a fixed point of B if and only if $\alpha \in B(\alpha)$ if and only if α is a mixed-strategy Nash equilibrium.

The graph of B has to be connected. Then there is at least one point on the fixpoint diagonal. $\square$

Nash's Theorem

Outline for the formal proof:

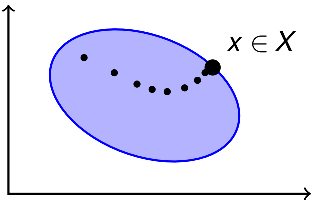
1. Review of necessary mathematical definitions
   $\rightsquigarrow$ Subsection "Definitions"
2. Statement of a fixpoint theorem used to prove Nash's theorem (without proof)
   $\rightsquigarrow$ Subsection "Kakutani's Fixpoint Theorem"
3. Proof of Nash's theorem using fixpoint theorem
   $\rightsquigarrow$ Subsection "Proof of Nash's Theorem"

Definition

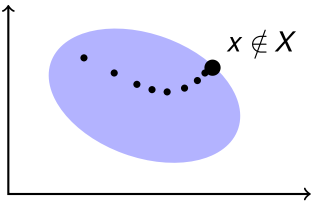
A set $X \subseteq \mathbb{R}^n$ is closed if X contains all its limit points, i. e., if $(x_k)_{k \in \mathbb{N}}$ is a sequence of elements in X and $\lim_{k \to \infty} x_k = x$, then also $x \in X$.

Example

Closed:



Not closed:



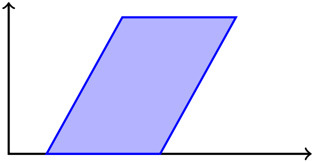
Definition

A set $X \subseteq \mathbb{R}^n$ is bounded if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

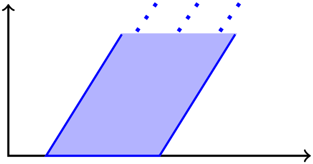
$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Example

Bounded:



Not bounded:



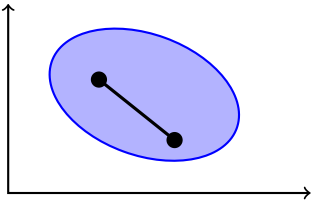
Definition

A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0,1]$,

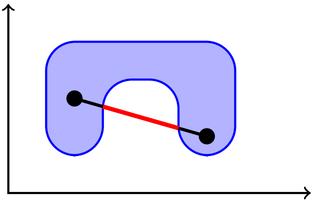
$$\lambda x + (1-\lambda)y \in X.$$

Example

Convex:



Not convex:



Definition

For a function $f : X \to 2^X$, the graph of f is the set

$$Graph(f) = \{(x,y) \mid x \in X,\ y \in f(x)\}.$$

Theorem (Kakutani's fixpoint theorem)

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \to 2^X$ be a function such that

- *for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and*
- *$Graph(f)$ is closed.*

Then there is an $x \in X$ with $x \in f(x)$, i. e., f has a fixpoint.

Proof.

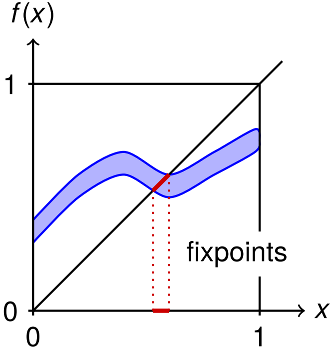
See Shizuo Kakutani, A generalization of Brouwer's fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232). □

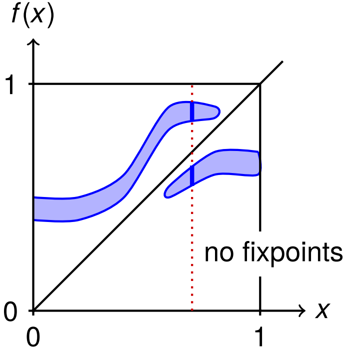
Example

Let $X = [0, 1]$.

Kakutani's theorem applicable:



Kakutani's theorem not applicable:



Proof.

Apply Kakutani's fixpoint theorem using $X = \mathscr{A} = \prod_{i \in N} \Delta(A_i)$ and $f = B$, where $B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i})$.

We have to show:

1. $\mathscr{A}$ is nonempty,
2. $\mathscr{A}$ is closed,
3. $\mathscr{A}$ is bounded,
4. $\mathscr{A}$ is convex,
5. $B(\alpha)$ is nonempty for all $\alpha \in \mathscr{A}$,
6. $B(\alpha)$ is convex for all $\alpha \in \mathscr{A}$, and
7. $Graph(B)$ is closed.

Proof (ctd.)

Some notation:

- Assume without loss of generality that $N = \{1, \dots, n\}$.
- A profile of mixed strategies can be written as a vector of $M = \sum_{i \in N} |A_i|$ real numbers in the interval $[0, 1]$ such that numbers for the same player add up to 1.

  For example, $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1(T) = 0.7$, $\alpha_1(M) = 0.0$, $\alpha_1(B) = 0.3$, $\alpha_2(L) = 0.4$, $\alpha_2(R) = 0.6$ can be seen as the vector

  $$(\underbrace{0.7,\ 0.0,\ 0.3}_{\alpha_1},\ \underbrace{0.4,\ 0.6}_{\alpha_2})$$

- This allows us to interpret the set $\mathscr{A}$ of mixed strategy profiles as a subset of $\mathbb{R}^M$.

Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0, \dots, 0}_{|A_1|-1 \text{ times}}, \dots, 1, \underbrace{0, \dots, 0}_{|A_n|-1 \text{ times}}).$$

2. $\mathscr{A}$ closed: Let $\alpha_1, \alpha_2, \dots$ be a sequence in $\mathscr{A}$ that converges to $\lim_{k\to\infty} \alpha_k = \alpha$. Suppose $\alpha \notin \mathscr{A}$. Then either there is some component of α that is less than zero or greater than one, or the components for some player i add up to a value other than one.

   Since α is a limit point, the same must hold for some α_k in the sequence. But then, $\alpha_k \notin \mathscr{A}$, a contradiction. Hence $\mathscr{A}$ is closed.

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Proof (ctd.)

3. $\mathscr{A}$ bounded: Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0,1]^M$.
4. $\mathscr{A}$ convex: Let $\alpha, \beta \in \mathscr{A}$ and $\lambda \in [0,1]$, and consider $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned}\min(\gamma) &= \min(\lambda\alpha + (1-\lambda)\beta) \\ &\geq \lambda \cdot \min(\alpha) + (1-\lambda) \cdot \min(\beta) \\ &\geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0,\end{aligned}$$

and similarly, $\max(\gamma) \leq 1$.

Hence, all entries in γ are still in $[0,1]$.

Proof (ctd.)

4 $\mathscr{A}$ convex (ctd.): Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the sections of α, β and γ, respectively, that determine the probability distribution for player *i*. Then

$$\begin{aligned}\sum \tilde{\gamma} &= \sum(\lambda\tilde{\alpha} + (1-\lambda)\tilde{\beta}) \\ &= \lambda \cdot \sum \tilde{\alpha} + (1-\lambda) \cdot \sum \tilde{\beta} \\ &= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.\end{aligned}$$

Hence, all probabilities for player *i* in γ still sum up to 1.

Altogether, $\gamma \in \mathscr{A}$, and therefore, $\mathscr{A}$ is convex.

Proof (ctd.)

5. $B(\alpha)$ nonempty: For a fixed α_{-i}, U_i is linear in the mixed strategies of player i, i. e., for $\beta_i, \gamma_i \in \Delta(A_i)$,

$$U_i\left(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i\right) = \lambda U_i\left(\alpha_{-i}, \beta_i\right) + (1-\lambda)U_i\left(\alpha_{-i}, \gamma_i\right) \tag{1}$$

for all $\lambda \in [0,1]$.

Hence, U_i is continous on $\Delta(A_i)$.

Continous functions on closed and bounded sets take their maximum in that set.

Therefore, $B_i(\alpha_{-i}) \neq \emptyset$ for all $i \in N$, and thus $B(\alpha) \neq \emptyset$.

Proof (ctd.)

6. $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex.
   To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.
   Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.
   With Equation (1), this implies
   $$\lambda \alpha_i' + (1 - \lambda)\alpha_i'' \in B_i(\alpha_{-i}).$$
   Hence, $B_i(\alpha_{-i})$ is convex.
7. *Graph*(B) closed: Let (α^k, β^k) be a convergent sequence in *Graph*(B) with $\lim_{k\to\infty}(\alpha^k, \beta^k) = (\alpha, \beta)$.
   So, $\alpha^k, \beta^k, \alpha, \beta \in \prod_{i\in N} \Delta(A_i)$ and $\beta^k \in B(\alpha^k)$.
   We need to show that $(\alpha, \beta) \in$ *Graph*(B), i. e., that $\beta \in B(\alpha)$.

Proof (ctd.)

7 *Graph*(*B*) closed (ctd.): It holds for all $i \in N$:

$$
\begin{aligned}
U_i\big(\alpha_{-i}, \beta_i\big) &\overset{\text{(D)}}{=} U_i\big(\lim_{k\to\infty}(\alpha^k_{-i}, \beta^k_i)\big) \\
&\overset{\text{(C)}}{=} \lim_{k\to\infty} U_i\big(\alpha^k_{-i}, \beta^k_i\big) \\
&\overset{\text{(B)}}{\geq} \lim_{k\to\infty} U_i\big(\alpha^k_{-i}, \beta'_i\big) \quad \text{for all } \beta'_i \in \Delta(A_i) \\
&\overset{\text{(C)}}{=} U_i\big(\lim_{k\to\infty} \alpha^k_{-i}, \beta'_i\big) \quad \text{for all } \beta'_i \in \Delta(A_i) \\
&\overset{\text{(D)}}{=} U_i\big(\alpha_{-i}, \beta'_i\big) \quad \text{for all } \beta'_i \in \Delta(A_i).
\end{aligned}
$$

(D): def. α_i, β_i; (C) continuity; (B) β^k_i best response to α^k_{-i}.

Proof (ctd.)

7. *Graph*(B) closed (ctd.): It follows that β_i is a best response to α_{-i} for all $i \in N$.
   Thus, $\beta \in B(\alpha)$ and finally $(\alpha, \beta) \in \mathit{Graph}(B)$.

Therefore, all requirements of Kakutani's fixpoint theorem are satisfied.

Applying Kakutani's theorem establishes the existence of a fixpoint of B, which is, by definition/construction, the same as a mixed-strategy Nash equilibrium. $\square$

3 Correlated Equilibria

Correlated Equilibria

Recall: There are three Nash equilibria in Bach or Stravinsky

- (B,B) with payoff profile $(2,1)$
- (S,S) with payoff profile $(1,2)$
- (α_1^*, α_2^*) with payoff profile $(2/3, 2/3)$ where
  - $\alpha_1^*(B) = 2/3$, $\alpha_1^*(S) = 1/3$,
  - $\alpha_2^*(B) = 1/3$, $\alpha_2^*(S) = 2/3$.

Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.

Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:

- If the coin shows heads, both play B.
- If the coin shows tails, both play S.

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: $(3/2, 3/2)$ instead of $(2/3, 2/3)$.

Observations and Information Partitions

We assume that observations are made based on a finite probability space (Ω, π), where Ω is a set of states and π is a probability measure on Ω.

Agents might not be able to distingush all states from each other. In order to model this, we assume for each player i an information partition $\mathscr{P}_i = \{P_{i1}, P_{i2}, \dots, P_{ik_i}\}$. This means that $\bigcup \mathscr{P}_i = \Omega$ for all i, and for all $P_j, P_k \in \mathscr{P}_i$ with $P_j \neq P_k$, we have $P_j \cap P_k = \emptyset$.

Example: $\Omega = \{x, y, z\}$, $\mathscr{P}_1 = \{\{x\}, \{y, z\}\}$, $\mathscr{P}_2 = \{\{x, y\}, \{z\}\}$.

We say that a function $f : \Omega \to X$ respects an information partition for player i if $f(\omega) = f(\omega')$ whenever $\omega, \omega' \in P_i$ for some $P_i \in \mathscr{P}_i$.

Example: f respects $\mathscr{P}_1$ if $f(y) = f(z)$.

Definition

A correlated equilibrium of a strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of

- a finite probability space (Ω, π),
- for each player $i \in N$ an information partition $\mathscr{P}_i$ of Ω,
- for each player $i \in N$ a function $\sigma_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (σ_i is player i's strategy)

such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (i.e. for every possible strategy of player i) we have

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)). \qquad (2)$$

Example

	L	R
T	$6,6$	$2,7$
B	$7,2$	$0,0$

Equilibria: (T,R) with $(2,7)$, (B,L) with $(7,2)$, and mixed $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with $(4+\frac{2}{3},4+\frac{2}{3})$.

Assume $\Omega = \{x,y,z\}, \pi(x) = \frac{1}{3}, \pi(y) = \frac{1}{3}, \pi(z) = \frac{1}{3}$.
Assume further $\mathscr{P}_1 = \{\{x\},\{y,z\}\}, \mathscr{P}_2 = \{\{x,y\},\{z\}\}$.
Set $\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T$ and $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$.

Then both player play optimally and get a payoff profile of $(5,5)$.

Proposition

For every mixed strategy Nash equilibrium α of a finite strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, there is a correlated equilibrium $\langle (\Omega, \pi), (\mathscr{P}_i), (\sigma_i) \rangle$ in which for each player i the distribution on A_i induced by σ_i is α_i.

This means that correlated equilibria are a generalization of Nash equilibria.

Proof

Proof.

Let $\Omega = A$ and define $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$. For each player i, let $a \in P$ and $b \in P$ for $P \in \mathscr{P}_i$ if $a_i = b_i$. Define $\sigma_i(a) = a_i$ for each $a \in A$.

Then $\langle(\Omega, \pi), (\mathscr{P}_i), (\sigma_i)\rangle$ is a correlated equilibrium since the left hand side of (2) is the Nash equilibrium payoff and for each player i at least as good any other strategy τ_i respecting the information partition. Further, the distribution induced by σ_i is α_i. □

Proposition

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. Any convex combination of correlated equilibirum payoff profiles of G is a correlated equilibirum payoff profile of G.

Proof idea: From given equilibria and weighting factors, create a new one by combining them orthogonally, using the weighting factors.

Proof.

Let $u^1, \ldots, u^K$ be the payoff profiles and let $(\lambda^1, \ldots, \lambda^K) \in \mathbb{R}^K$ with $\lambda^l \geq 0$ and $\sum_{l=1}^{K} \lambda^l = 1$. For each l let $\langle(\Omega^l, \pi^l), (\mathscr{P}_i^l), (\sigma_i^l)\rangle$

be a correlated equilibrium generating payoff u^l. Wlog. assume all Ω^l's are disjoint.

Now we define a correlated equilibrium generating the payoff $\sum_{l=1}^{K} \lambda^l u^l$. Let $\Omega = \bigcup_l \Omega^l$. For any $\omega \in \Omega$ define $\pi(\omega) = \lambda^l \pi^l(\omega)$ where l is such that $\omega \in \Omega^l$. For each $i \in N$ let $\mathscr{P}_i = \bigcup_l \mathscr{P}_i^l$ and set $\sigma_i(\omega) = \sigma_i^l(\omega)$ where l is such that $\omega \in \Omega^l$.

$\square$

Basically, first throw a dice for which CE to go for, then proceed in this CE.

4 Summary

Summary

- Mixed strategies allow randomization.
- Characterization of mixed-strategy Nash equilibria: players only play best responses with positive probability (support lemma).
- Nash's Theorem: Every finite strategic game has a mixed-strategy Nash equilibrium.
- Correlated equilibria can lead to higher payoffs.
- All Nash equilibria are correlated equilibria, but not *vice versa*.