Game Theory

2. Strategic Games

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

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Preliminaries and Examples

Strategic Games

Definition (Strategic game)

A strategic game is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where

- a nonempty finite set N of players,
- for each player $i \in N$, a nonempty set A_i of actions (or strategies), and
- for each player $i \in N$, a payoff function $u_i : A \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$.

A strategic game G is called finite if A is finite.

A strategy profile is a tuple $a = (a_1, \ldots, a_{|N|}) \in A$.

Strategic Games

We can describe finite strategic games using payoff matrices.

Example: Two-player game where player 1 has actions T and B, and player 2 has actions L and R, with payoff matrix

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{player 2}} \\
 & & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{R} \\ \cline{3-4}
\text{player 1} & T & w_1, w_2 & x_1, x_2 \\ \cline{3-4}
 & B & y_1, y_2 & z_1, z_2 \\ \cline{3-4}
\end{array}$$

Read: **If** player 1 plays T and player 2 plays L
then player 1 gets payoff w_1 and player 2 gets payoff w_2, etc.

Prisoner’s Dilemma

Example (Prisoner’s Dilemma (informally))

Two prisoners are interrogated separately, and have the options to either cooperate (C) with their fellow prisoner and stay silent, or defect (D) and accuse the fellow prisoner of the crime.

Possible outcomes:

- Both cooperate: no hard evidence against either of them, only short prison sentences for both.
- One cooperates, the other defects: the defecting prisoner is set free immediately, and the cooperating prisoner gets a very long prison sentence.
- Both confess: both get medium-length prison sentences.

Prisoner's Dilemma

Example (Prisoner's Dilemma (payoff matrix))

Strategies $A_1 = A_2 = \{C, D\}$.

		player 2	
		C	D
player 1	C	$3, 3$	$0, 4$
	D	$4, 0$	$1, 1$

Hawk and Dove

An anti-coordination game:

Example (Hawk and Dove (informally))

In a fight for resources two players can behave either like a dove (*D*), yielding, or like a hawk (*H*), attacking.

Possible outcomes:

- Both players behave like doves: both players share the benefit.
- A hawk meets a dove: the hawk wins and gets the bigger part.
- Both players behave like hawks: the benefit gets lost completely because they will fight each other.

Hawk and Dove

Example (Hawk and Dove (payoff matrix))

Strategies $A_1 = A_2 = \{D, H\}$.

$$
\begin{array}{ccc|c|c|}
 & & \multicolumn{2}{c}{\text{player 2}} \\
 & & \multicolumn{1}{c}{D} & \multicolumn{1}{c}{H} \\
\cline{3-4}
\text{player 1} & D & 3,3 & 1,4 \\
\cline{3-4}
 & H & 4,1 & 0,0 \\
\cline{3-4}
\end{array}
$$

Matching Pennies

A strictly competitive game:

Example (Matching Pennies (informally))

Two players can choose either heads (*H*) or tails (*T*) of a coin.

Possible outcomes:

- Both players make the same choice: player 1 receives one Euro from player 2.
- The players make different choices: player 2 receives one Euro from player 1.

Matching Pennies

Example (Matching Pennies (payoff matrix))

Strategies $A_1 = A_2 = \{H, T\}$.

		player 2	
		H	*T*
player 1	*H*	$1, -1$	$-1, \ \ 1$
	T	$-1, \ \ 1$	$1, -1$

Bach or Stravinsky (aka Battle of the Sexes)

A coordination game:

Example (Bach or Stravinsky (informally))

Two persons, one of whom prefers Bach whereas the other prefers Stravinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite one. Let *B* be the action of going to the Bach concert and *S* the action of going to the Stravinsky concert.

Possible outcomes:

- Both players make the same choice: the player whose preferred option is chosen gets high payoff, the other player gets medium payoff.
- The players make different choices: they both get zero payoff.

Bach or Stravinsky (aka Battle of the Sexes)

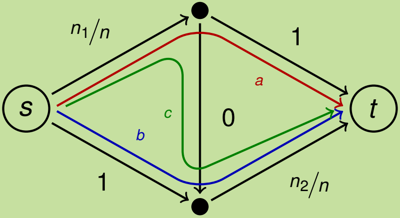
Example (Bach or Stravinsky (payoff matrix))

Strategies $A_1 = A_2 = \{B, S\}$.

		Stravinsky enthusiast	
		B	S
Bach enthusiast	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

Congestion Game

Example (A congestion game)



		player 2		
		a	*b*	*c*
	a	$-2, -2$	$-1.5, -1.5$	$-2, -1.5$
player 1	*b*	$-1.5, -1.5$	$-2, -2$	$-2, -1.5$
	c	$-1.5, -2$	$-1.5, -2$	$-2, -2$

Solution Concepts and Notation

Solution Concepts and Notation

Question: What is a “solution” of a strategic game?

Answer:

- A strategy profile where all players play strategies that are rational (i. e., in some sense optimal).
- Note: There are different ways of making the above item precise (different solution concepts).
- A solution concept is a formal rule for predicting how a game will be played.

In the following, we will consider some solution concepts:

- Iterated dominance
- Nash equilibrium
- (Subgame-perfect equilibrium)

Solution Concepts and Notation

Notation: we want to write down strategy profiles where one player's strategy is removed or replaced.

Let $a = (a_1, \dots, a_{|N|}) \in A = \prod_{i \in N} A_i$ be a strategy profile.

We write:

- $A_{-i} := \prod_{j \in N \setminus \{i\}} A_j$,
- $a_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{|N|})$, and
- $(a_{-i}, a'_i) := (a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_{|N|})$.

Example

Let $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $A_3 = \{X, Y, Z\}$, and $a := (T, R, Z)$.
Then $a_{-1} = (R, Z)$, $a_{-2} = (T, Z)$, $a_{-3} = (T, R)$.
Moreover, $(a_{-2}, L) = (T, L, Z)$.

Dominated Strategies

Strictly Dominated Strategies

Question: What strategy should an agent avoid?

One answer:

- Eliminate all obviously irrational strategies.
- A strategie is obviously irrational if there is another strategy that is always better, no matter what the other players do.

Strictly Dominated Strategies

Definition (Strictly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called strictly dominated in G if there is a strategy $a_i^+ \in A_i$ such that for all strategy profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ strictly dominates a_i.

If $a_i^+ \in A_i$ strictly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ strictly dominant in G.

Remark: Playing strictly dominated strategies is irrational.

Strictly Dominated Strategies

This suggest a solution concept:
iterative elimination of strictly dominated strategies:

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while some strictly dominated strategy is left:
    eliminate some strictly dominated strategy
if a unique strategy profile remains:
    this unique profile is the solution
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Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

player 2

		C	D
player 1	C	$3, 3$	$0, 4$
	D	$4, 0$	$1, 1$

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

player 2

		C	D
player 1	~~C~~	~~3, 3~~	~~0, 4~~
	D	4, 0	1, 1

- Step 1: eliminate row C (strictly dominated by row D)

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

player 2

player 1

$$\begin{array}{c|c|c}
 & \xcancel{C} & D \\ \hline
\xcancel{C} & \xcancel{3,3} & \xcancel{0,4} \\ \hline
D & \xcancel{4,0} & 1,1
\end{array}$$

- Step 1: eliminate row *C* (strictly dominated by row *D*)
- Step 2: eliminate column *C* (strictly dominated by col. *D*)

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		~~*C*~~	*D*
player 1	~~*C*~~	~~3, 3~~	~~0, 4~~
	D	~~4, 0~~	1, 1

- Step 1: eliminate row *C* (strictly dominated by row *D*)
- Step 2: eliminate column *C* (strictly dominated by col. *D*)

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

player 2

		L	*R*
	T	2, 1	0, 0
player 1	*M*	1, 2	2, 1
	B	0, 0	1, 1

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	*R*
	T	2, 1	0, 0
player 1	*M*	1, 2	2, 1
	~~*B*~~	~~0, 0~~	~~1, 1~~

- Step 1: eliminate row *B* (strictly dominated by row *M*)

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	~~*R*~~
	T	2, 1	~~0, 0~~
player 1	*M*	1, 2	~~2, 1~~
	~~*B*~~	~~0, 0~~	~~1, 1~~

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

player 2

		L	~~*R*~~
	T	2, 1	~~0, 0~~
player 1	~~*M*~~	~~1, 2~~	~~2, 1~~
	~~*B*~~	~~0, 0~~	~~1, 1~~

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)
- Step 3: eliminate row *M* (strictly dominated by row *T*)

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	~~*R*~~
	T	2, 1	~~0, 0~~
player 1	~~*M*~~	~~1, 2~~	~~2, 1~~
	~~*B*~~	~~0, 0~~	~~1, 1~~

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)
- Step 3: eliminate row *M* (strictly dominated by row *T*)

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

		Stravinsky enthusiast	
		B	S
Bach enthusiast	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

Stravinsky enthusiast

Bach enthusiast		B	S
	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

- No strictly dominated strategies.
- All strategies survive iterative elimination of strictly dominated strategies.
- All strategies rationalizable.

Strictly Dominated Strategies

Remark

Strict dominance between actions is rather rare.
We should identify more constraints on “solutions”, better solution concepts.

Proposition

The result of iterative elimination of strictly dominated strategies is unique, i. e., independent of the elimination order.

Proof.

Homework. □

Weakly Dominated Strategies

Definition (Weakly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called weakly dominated in G if there is a strategy $a_i^+ \in A_i$ such that for all profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+)$$

and that for at least one profile $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ weakly dominates a_i.

If $a_i^+ \in A_i$ weakly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ weakly dominant in G.

Weakly Dominated Strategies

What about
iterative elimination of weakly dominated strategies
as a solution concept?

Let’s see what happens.

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	R
	T	$2,1$	$0,0$
player 1	M	$2,1$	$1,1$
	B	$0,0$	$1,1$

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	*R*
	T	2, 1	0, 0
player 1	*M*	2, 1	1, 1
	~~*B*~~	~~0, 0~~	~~1, 1~~

- Step 1: eliminate row B (weakly dominated by row M, $u_1(M,L) = 2 > 0 = u_1(B,L)$ and $u_1(M,R) = 1 = u_1(B,R)$)

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	~~R~~
	T	$2,1$	~~$0,0$~~
player 1	M	$2,1$	~~$1,1$~~
	~~B~~	~~$0,0$~~	~~$1,1$~~

- Step 1: eliminate row B (weakly dominated by row M, $u_1(M,L) = 2 > 0 = u_1(B,L)$ and $u_1(M,R) = 1 = u_1(B,R)$)
- Step 2: eliminate column R (weakly dominated by col. L)

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	~~*R*~~
	T	2, 1	~~0, 0~~
player 1	*M*	2, 1	~~1, 1~~
	~~*B*~~	~~0, 0~~	~~1, 1~~

- Step 1: eliminate row B (weakly dominated by row M, $u_1(M,L) = 2 > 0 = u_1(B,L)$ and $u_1(M,R) = 1 = u_1(B,R)$)
- Step 2: eliminate column R (weakly dominated by col. L)

Here, two solution profiles remain.

Weakly Dominated Strategies

Iterative elimination of weakly dominated strategies:

- leads to smaller games,
- can also lead to situations where only a single solution remains,
- but: the result can depend on the elimination order! (see example on next slide)

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

player 2

		L	R
	T	$2,1$	$0,0$
player 1	M	$2,1$	$1,1$
	B	$0,0$	$1,1$

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	*R*
	~~*T*~~	~~$2, 1$~~	~~$0, 0$~~
player 1	*M*	$2, 1$	$1, 1$
	B	$0, 0$	$1, 1$

- Step 1: eliminate row *T* (weakly dominated by row *M*)

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		~~*L*~~	*R*
	~~*T*~~	~~2, 1~~	~~0, 0~~
player 1	*M*	~~2, 1~~	1, 1
	B	~~0, 0~~	1, 1

- Step 1: eliminate row *T* (weakly dominated by row *M*)
- Step 2: eliminate column *L* (weakly dominated by col. *R*)

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		~~L~~	R
	~~T~~	~~2, 1~~	~~0, 0~~
player 1	M	~~2, 1~~	1, 1
	B	~~0, 0~~	1, 1

- Step 1: eliminate row T (weakly dominated by row M)
- Step 2: eliminate column L (weakly dominated by col. R)

Different elimination order, different result,
even different payoffs (1, 1 vs. 2, 1)!

Nash Equilibria

Nash Equilibria

Question: Which strategy profiles are stable?

Possible answer:

- Strategy profiles where no player benefits from playing a different strategy
- Equivalently: Strategy profiles where every player's strategy is a best response to the other players' strategies

Such strategy profiles are called Nash equilibria, one of the most-used solution concepts in game theory.

Remark: In following examples, for non-Nash equilibria, only one possible profitable deviation is shown (even if there are more).

Nash Equilibria

Preliminaries and Examples

Solution Concepts and Notation

Dominated Strategies

Nash Equilibria

Definitions and Examples

Example: Sealed-Bid Auctions

Iterative Elimination and Nash Equilibria

Zero-Sum Games

Summary

Definition (Nash equilibrium)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$,

$$u_i(a^*) \geq u_i(a^*_{-i}, a_i) \qquad \text{for all } a_i \in A_i.$$

Nash Equilibria

Remark: There is an alternative definition of Nash equilibria (which we consider because it gives us a slightly different perspective on Nash equilibria).

Definition (Best response)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game, $i \in N$ a player, and $a_{-i} \in A_{-i}$ a strategy profile of the players other than i. Then a strategy $a_i \in A_i$ is a best response of player i to a_{-i} if

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a_i') \qquad \text{for all } a_i' \in A_i.$$

We write $B_i(a_{-i})$ for the set of best responses of player i to a_{-i}.

For a strategy profile $a \in A$, we write $B(a) = \prod_{i \in N} B_i(a_{-i})$.

Nash Equilibria

Definition (Nash equilibrium, alternative 1)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$, $a_i^* \in B_i(a_{-i}^*)$.

Definition (Nash equilibrium, alternative 2)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that $a^* \in B(a^*)$.

Proposition

The three definitions of Nash equilibria are equivalent.

Proof.

Homework. □

Nash Equilibria

Example (Nash Equilibria in the Prisoner's Dilemma)

		player 2	
		C	D
player 1	C	$3,3$	$0,4$
	D	$4,0$	$1,1$

- (C,C): No Nash equilibrium (player 1: $C \to D$)
- (C,D): No Nash equilibrium (player 1: $C \to D$)
- (D,C): No Nash equilibrium (player 2: $C \to D$)
- (D,D): Nash equilibrium!

Nash Equilibria

Example (Nash Equilibria in Hawk and Dove)

		player 2	
		D	*H*
player 1	*D*	3,3	1,4
	H	4,1	0,0

- (D,D): No Nash equilibrium (player 1: $D \rightarrow H$)
- (D,H): Nash equilibrium!
- (H,D): Nash equilibrium!
- (H,H): No Nash equilibrium (player 1: $H \rightarrow D$)

Nash Equilibria

Example (Nash Equilibria in Matching Pennies)

		player 2	
		H	T
player 1	H	$1, -1$	$-1, \ 1$
	T	$-1, \ 1$	$1, -1$

- (H, H): No Nash equilibrium (player 2: $H \to T$)
- (H, T): No Nash equilibrium (player 1: $H \to T$)
- (T, H): No Nash equilibrium (player 1: $T \to H$)
- (T, T): No Nash equilibrium (player 2: $T \to H$)

Nash Equilibria

Example (Nash Equilibria in Bach or Stravinsky)

Stravinsky enthusiast

		B	S
Bach enthusiast	B	$2,1$	$0,0$
	S	$0,0$	$1,2$

- (B,B): Nash equilibrium!
- (B,S): No Nash equilibrium (player 1: $B \to S$)
- (S,B): No Nash equilibrium (player 2: $S \to B$)
- (S,S): Nash equilibrium!

Example: Sealed-Bid Auctions

We consider a slightly larger example: sealed-bid auctions

Setting:

- An object has to be assigned to a winning bidder in exchange for a payment.
- For each player ("bidder") $i = 1, \dots, n$, let v_i be the private value that bidder i assigns to the object.
  (We assume that $v_1 > v_2 > \cdots > v_n > 0$.)
- The bidders simultaneously give their bids $b_i \geq 0$, $i = 1, \dots, n$.
- The object is given to the bidder i with the highest bid b_i.
  (Ties are broken in favor of bidders with lower index, i.e., if $b_i = b_j$ are the highest bids, then bidder i will win iff $i < j$.)

Example: Sealed-Bid Auctions

Question: What should the winning bidder have to pay?

One possible answer: The highest bid.

Definition (First-price sealed-bid auction)

- $N = \{1, \ldots, n\}$ with $v_1 > v_2 > \cdots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - b_i & \text{otherwise} \end{cases}$
  where $b = (b_1, \ldots, b_n)$.

Example: Sealed-Bid Auctions

Example (First-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$v_1 = 100, \qquad v_2 = 80, \qquad v_3 = 53,$$
$$b_1 = 90, \qquad b_2 = 85, \qquad b_3 = 45.$$

Observations:

- Bidder 1 wins, pays 90, gets utility $u_1(b) = v_1 - b_1 = 100 - 90 = 10$.
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding $b_1' = 85$ instead. Then $u_1(b_{-1}, b_1') = v_1 - b_1' = 100 - 85 = 15$.

Example: Sealed-Bid Auctions

Question: How to avoid untruthful bidding and incentivize truthful revelation of private valuations?

Different answer to question about payments: Winner pays the second-highest bid.

Definition (Second-price sealed-bid auction)

- $N = \{1, \dots, n\}$ with $v_1 > v_2 > \cdots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - \max b_{-i} & \text{otherwise} \end{cases}$ where $b = (b_1, \dots, b_n)$.

Example: Sealed-Bid Auctions

Example (Second-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$v_1 = 100, \quad v_2 = 80, \quad v_3 = 53,$$
$$b_1 = 90, \quad b_2 = 85, \quad b_3 = 45.$$

Observations:

- Bidder 1 wins, pays 85, gets utility $u_1(b) = v_1 - b_2 = 100 - 85 = 15$.
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess the other bidders' private valuations.

Example: Sealed-Bid Auctions

Proposition

In a second-price sealed-bid auction, bidding ones own valuation, $b_i^+ = v_i$, is a weakly dominant strategy.

Proof.

We have to show that b_i^+ weakly dominates every other strategy b_i of player i.

For that, it suffices to show that

1. for all $b_i \in A_i$, we have
   $u_i(b_{-i}, b_i^+) \geq u_i(b_{-i}, b_i)$ for all $b_{-i} \in A_{-i}$, and that
2. for all $b_i \in A_i$, we have
   $u_i(b_{-i}, b_i^+) > u_i(b_{-i}, b_i)$ for at least one $b_{-i} \in A_{-i}$.

Example: Sealed-Bid Auctions

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2. for all $b_i \in A_i$, we have
   $u_i(b_{-i}, b_i^+) > u_i(b_{-i}, b_i)$ for at least one $b_{-i} \in A_{-i}$.

Example: Sealed-Bid Auctions

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For that, it suffices to show that

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2. for all $b_i \in A_i$, we have
   $u_i(b_{-i}, b_i^+) > u_i(b_{-i}, b_i)$ for at least one $b_{-i} \in A_{-i}$.

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) [regardless of what the other bidders do,
b_i^+ is always a best response]:

- Case I) bidder *i* wins:
  bidder *i* pays $\max b_{-i} \leq v_i$, gets $u_i(b_{-i}, b_i^+) \geq 0$.
  - Case I.a) bidder *i* decreases bid:
    this does not help, since he might still win and pay the same as before, or lose and get utility 0.
  - Case I.b) bidder *i* increases bid:
    bidder *i* still wins and pays the same as before.

Example: Sealed-Bid Auctions

Proof (ctd.)

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Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) (ctd.):

- Case II) bidder *i* loses:
  bidder *i* pays nothing, gets $u_i(b_{-i}, b_i^+) = 0$.
  - Case II.a) bidder *i* decreases bid:
    bidder *i* still loses and gets utility 0.
  - Case II.b) bidder *i* increases bid:
    either bidder *i* still loses and gets utility 0, or becomes the winner and pays more than the object is worth to him, leading to a negative utility.

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Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) [for each alternative b_i to b_i^+, there is an opponent profile b_{-i} against which b_i^+ is strictly better than b_i]:

Let b_i be some strategy other than b_i^+.

- Case I) $b_i < b_i^+$:
  Consider b_{-i} with $b_i < \max b_{-i} < b_i^+$.
  With b_i, bidder i does not win any more, i. e., we have $u_i(b_{-i}, b_i^+) > 0 = u_i(b_{-i}, b_i)$.
- Case II) $b_i > b_i^+$:
  Consider b_{-i} with $b_i > \max b_{-i} > b_i^+$.
  With b_i, bidder i overbids and pays more than the object is worth to him, i. e., we have $u_i(b_{-i}, b_i^+) = 0 > u_i(b_{-i}, b_i)$.

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☐

Example: Sealed-Bid Auctions

Proposition

Profiles of weakly dominant strategies are Nash equilibria.

Proof.

Homework. ☐

Proposition

In a second-price sealed-bid auction, if all bidders bid their true valuations, this is a Nash equilibrium.

Proof.

Follows immediately from the previous two propositions. ☐

Remark: This is not the only Nash equilibrium in second-price sealed-bid auctions, though.

Iterative Elimination and Nash Equilibria

Motivation: We have seen two different solution concepts,

- Surviving iterative elimination of (strictly) dominated strategies and
- Nash equilibria.

Obvious question: Is there any relationship between the two?

Answer: Yes, Nash equilibria refine the concept of iterative elimination of strictly dominated strategies. We will formalize this on the next slides.

Iterative Elimination and Nash Equilibria

Lemma (preservation of Nash equilibria)

Let G and G' be two strategic games where G' is obtained from G by elimination of one strictly dominated strategy. Then a strategy profile a^* is a Nash equilibrium of G if and only if it is Nash equilibrium of G'.

Proof.

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ and $G' = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$.

Let a'_i be the eliminated strategy.
Then there is a strategy a^+_i such that for all $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a'_i) < u_i(a_{-i}, a^+_i). \qquad (1)$$

Iterative Elimination and Nash Equilibria

Proof (ctd.)

“$\Rightarrow$”: Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

  For player i, action a_i^* is a best response to a_{-i}^*, and in particular at least as good a response as a_i^+:

  $$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

  With (1) $u_i(a_{-i}, a_i^+) > u_i(a_{-i}, a_i')$, we get $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i')$ and hence $a_i^* \neq a_i'$.

  Thus, the Nash equilibrium strategy a_i^* is not eliminated.

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Iterative Elimination and Nash Equilibria

Proof (ctd.)

“$\Rightarrow$” (ctd.):

- **Best responses remain best responses:** For all players $j \in N$, a_j^* is a best response to a_{-j}^* in G. Since in G', no potentially better responses are introduced ($A_j' \subseteq A_j$) and the payoffs are unchanged, this also holds in G'.

Hence, a^* is also a Nash equilibrium of G'.

“$\Leftarrow$”: Let a^* be a Nash equilibrium of G'.

- For player $j \neq i$: a_j^* is a best response to a_{-j}^* in G as well, since the responses available to player j in G and G' are the same.

Iterative Elimination and Nash Equilibria

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Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Leftarrow$" (ctd.):

- For player i: Since $A_i = A_i' \cup \{a_i\}$ and a_i^* is a best response to a_{-i}^* among the strategies in A_i', it suffices to show that a_i is no better response.

  Because a^* is a Nash equilibrium in G' and a_i^+ is a strategy in A_i', we have $u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+)$.

  Since a_i^+ strictly dominates a_i, we have $u_i(a_{-i}^*, a_i^+) > u_i(a_{-i}^*, a_i)$, and hence $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i)$.

  Therefore, a_i cannot be a better response to a_{-i}^* than a_i^*.

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Iterative Elimination and Nash Equilibria

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Hence, a^* is also a Nash equilibrium of G. ☐

Iterative Elimination and Nash Equilibria

Corollary

If iterative elimination of strictly dominated strategies results in a *unique* strategy profile a^*, then a^* is the unique Nash equilibrium of the original game.

Proof.

Assume that a^* is the unique remaining strategy profile. By definition, a^* must be a Nash equilibrium of the remaining game.

We can inductively apply the previous lemma (preservation of Nash equilibria) and see that a^* (and no other strategy profile) must have been a Nash equilibrium before the last elimination step, and before that step, …, and in the original game. ☐

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Zero-Sum Games

Playing it Safe (in Two-Player Games)

Motivation: What happens if both players try to “play it safe”?

Question: What does it even mean to “play it safe”?

Answer: Choose a strategy that guarantees the highest worst-case payoff.

Playing it Safe (in Two-Player Games)

Example

player 2

		L	R
	T	$2,1$	$2,-20$
player 1	M	$3,0$	$-10,\quad 1$
	B	$-100,2$	$3,\quad 3$

Worst-case payoff for player 1:

- if playing T: 2
- if playing M: -10
- if playing B: -100

$\rightsquigarrow$ play T.

Worst-case payoff for player 2:

- if playing L: 0
- if playing R: -20

$\rightsquigarrow$ play L.

However: Unlike (B,R), the profile (T,L) is not a Nash equilibrium.

Playing it Safe (in Two-Player Games)

Example

player 2

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player 1	T	$2, 1$	$2, -20$
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However: Unlike (B, R), the profile (T, L) is not a Nash equilibrium.

Playing it Safe (in Two-Player Games)

Observation: In general, pairs of maximinimizers, like (T, L) in the example above, are not the same as Nash equilibria.

Claim: However, in zero-sum games, pairs of maximinimizers and Nash equilibria are essentially the same.

(Tiny restriction: This does not hold if the considered game has no Nash equilibrium at all, because unlike Nash equilibria, pairs of maximinimizers always exist.)

Reason (intuitively): In zero-sum games, the worst-case assumption that the other player tries to harm you as much as possible is justified, because harming the other is the same as maximizing ones own payoff. Playing it safe is rational.

Zero-Sum Games

Definition (Zero-sum game)

A zero-sum game is a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and

$$u_1(a) = -u_2(a)$$

for all $a \in A$.

Example (Matching Pennies as a zero-sum game)

player 2

		H	T
player 1	H	$1, -1$	$-1, \ \ 1$
	T	$-1, \ \ 1$	$1, -1$

Maximinimizers

Definition (Maximinimizer)

Let $G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game.

An action $x^* \in A_1$ is called maximinimizer for player 1 in G if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \qquad \text{for all } x \in A_1,$$

and $y^* \in A_2$ is called maximinimizer for player 2 in G if

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \qquad \text{for all } y \in A_2.$$

Maximinimizers

Example (Zero-sum game with three actions each)

player 2

		L	C	R
	T	$8, -8$	$3, -3$	$-6, \quad 6$
player 1	M	$2, -2$	$-1, \quad 1$	$3, -3$
	B	$-6, \quad 6$	$4, -4$	$8, -8$

Guaranteed worst-case payoffs:

- T: -6, M: -1, B: -6 ⇝ maximinimizer M
- L: -8, C: -4, R: -8 ⇝ maximinimizer C

⇝ pair of maximinimizers (M, C) with payoffs $(-1, 1)$
(not a Nash equilibrium; this game has no Nash equilibrium.)

Maximinimizers

Example (Maximinimization vs. minimaximization)

		player 2	
		L	*R*
player 1	*T*	$1, -1$	$2, -2$
	B	$-2, \ \ 2$	$-4, \ \ 4$

Worst-case payoffs (player 2):

- *L*: -1, *R*: -2
- Maximize: -1

Best-case payoffs (player 1):

- *L*: +1, *R*: +2
- Minimize: +1

Observation: Results identical up to different sign.

Maximinimizers

Lemma

Let $G = \langle \{1,2\}, (A_i)_{i\in N}, (u_i)_{i\in N} \rangle$ *be a zero-sum game. Then*

$$\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2} \max_{x\in A_1} u_1(x,y). \tag{2}$$

Proof.

For any real-valued function f, we have

$$\min_z -f(z) = -\max_z f(z). \tag{3}$$

Maximinimizers

Proof (ctd.)

Thus, for all $y \in A_2$,

$$
\begin{aligned}
-\min_{y \in A_2} \max_{x \in A_1} u_1(x,y) &\overset{(3)}{=} \max_{y \in A_2} -\max_{x \in A_1} u_1(x,y) \\
&\overset{(3)}{=} \max_{y \in A_2} \min_{x \in A_1} -u_1(x,y) \\
&\overset{\text{ZS}}{=} \max_{y \in A_2} \min_{x \in A_1} u_2(x,y).
\end{aligned}
$$

□

Nash Equilibria in Zero-Sum Games

Now, we are ready to prove our
main theorem about zero-sum games and Nash equilibria.

In zero-sum games:

1. Every Nash equilibrium is a pair of maximinimizers.
2. All Nash equilibria have the same payoffs.
3. If there is at least one Nash equilibrium, then every pair of maximinimizers is a Nash equilibrium.

Nash Equilibria in Zero-Sum Games

Theorem (Maximinimizer theorem)

Let $G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ *be a zero-sum game. Then:*

1. *If* (x^*, y^*) *is a Nash equilibrium of G, then* x^* *and* y^* *are maximinimizers for player 1 and player 2, respectively.*
2. *If* (x^*, y^*) *is a Nash equilibrium of G, then*

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) = u_1(x^*, y^*).$$

3. *If* $\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$, *and* x^* *and* y^* *maximinimizers of player 1 and player 2 respectively, then* (x^*, y^*) *is a Nash equilibrium.*

Nash Equilibria in Zero-Sum Games

Proof.

1. Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this implies

$$u_1(x^*, y^*) \leq u_1(x^*, y) \quad \text{for all } y \in A_2.$$

Thus

$$u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{4}$$

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Nash Equilibria in Zero-Sum Games

Proof (ctd.)

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   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

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   This implies

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Nash Equilibria in Zero-Sum Games

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   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

   $$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

   Hence

   $$u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).$$

   This implies

   $$u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \qquad (5)$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1. (ctd.)

   Inequalities (4) and (5) together imply that

   $$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{6}$$

   Thus, x^* is a maximinimizer for player 1.

   Similarly, we can show that y^* is a maximinimizer for player 2:

   $$u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \tag{7}$$

Nash Equilibria in Zero-Sum Games

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Nash Equilibria in Zero-Sum Games

Proof (ctd.)

2 We only need to put things together:

$$
\begin{aligned}
\max_{x\in A_1} \min_{y\in A_2} u_1(x,y) &\overset{(6)}{=} u_1(x^*,y^*) \\
&\overset{\text{ZS}}{=} -u_2(x^*,y^*) \\
&\overset{(7)}{=} -\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) \\
&\overset{(2)}{=} \min_{y\in A_2} \max_{x\in A_1} u_1(x,y).
\end{aligned}
$$

In particular, it follows that all Nash equilibria share the same payoff profile.

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3. Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*. \qquad (8)$$

With Equation (2) from the previous lemma, we get

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -v^*. \qquad (9)$$

With x^* and y^* being maximinimizers, (8) and (9) imply

$$u_1(x^*,y) \geq v^* \quad \text{for all } y \in A_2, \text{ and} \qquad (10)$$
$$u_2(x,y^*) \geq -v^* \quad \text{for all } x \in A_1. \qquad (11)$$

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Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3 (ctd.)

Special cases of (10) and (11) for $x = x^*$ and $y = y^*$:

$$u_1(x^*, y^*) \geq v^* \qquad \text{and} \qquad u_2(x^*, y^*) \geq -v^*.$$

With $u_1 = -u_2$, the latter is equivalent to $u_1(x^*, y^*) \leq v^*$, which gives us

$$u_1(x^*, y^*) = v^*. \tag{12}$$

Nash Equilibria in Zero-Sum Games

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Plugging (12) into the right-hand side of (10) gives us

$$u_1(x^*, y) \geq u_1(x^*, y^*) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this is equivalent to

$$u_2(x^*, y) \leq u_2(x^*, y^*) \quad \text{for all } y \in A_2.$$

In other words, y^* is a best response to x^*.

Nash Equilibria in Zero-Sum Games

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Proof (ctd.)

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Similarly, we can plug (12) into the right-hand side of (11) and obtain

$$u_2(x, y^*) \geq -u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

Again using $u_1 = -u_2$, this is equivalent to

$$u_1(x, y^*) \leq u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

In words, x^* is also a best response to y^*.

Hence, (x^*, y^*) is a Nash equilibrium. ☐

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In words, x^* is also a best response to y^*.

Hence, (x^*, y^*) is a Nash equilibrium.

□

Nash Equilibria in Zero-Sum Games

Corollary

Let $G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game, and let (x_1^*, y_1^*) and (x_2^*, y_2^*) be two Nash equilibria of G.

Then (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G.

In other words: Nash equilibria of zero-sum games can be arbitrarily recombined.

Nash Equilibria in Zero-Sum Games

Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

With part (2) of the maximinimizer theorem, we get that $\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$.

With this equality, with x_1^*, x_2^*, y_1^*, and y_2^* all being maximinimizers, and with part (3) of the maximinimizer theorem, we get that (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G. ☐

Nash Equilibria in Zero-Sum Games

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Nash Equilibria in Zero-Sum Games

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With this equality, with x_1^*, x_2^*, y_1^*, and y_2^* all being maximinimizers, and with part (3) of the maximinimizer theorem, we get that (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G. □

Summary

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- Strategic games are one-shot games of finitely many players with given action sets and payoff functions. Players have perfect information.
- Solution concepts: survival of iterative elimination of strictly dominated strategies, Nash equilibria.
- Relation between solution concepts: Nash equilibria always survive iterative elimination of strictly dominated strategies.
- In zero-sum games, one player's gain is the other player's loss. Thus, playing it safe is rational. Relevant concept: maximinimizers.
- Relation to Nash equilibria: In zero-sum games, Nash equilibria are pairs of maximinimizers, and, if at least one Nash equilibrium exists, pairs of maximinimizers are also Nash equilibria.