

Game Theory

2. Strategic Games

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1 Preliminaries and Examples



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Definition (Strategic game)

A **strategic game** is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where

- a nonempty finite set N of **players**,
- for each player $i \in N$, a nonempty set A_i of **actions** (or **strategies**), and
- for each player $i \in N$, a **payoff function** $u_i : A \rightarrow \mathbb{R}$, where $A = \prod_{i \in N} A_i$.

A strategic game G is called finite if A is finite.

A **strategy profile** is a tuple $a = (a_1, \dots, a_{|N|}) \in A$.

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We can describe finite strategic games using **payoff matrices**.

Example: Two-player game where player 1 has actions T and B , and player 2 has actions L and R , with payoff matrix

		player 2	
		L	R
player 1	T	w_1, w_2	x_1, x_2
	B	y_1, y_2	z_1, z_2

Read: If player 1 plays T and player 2 plays L then player 1 gets payoff w_1 and player 2 gets payoff w_2 , etc.

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Example (Prisoner's Dilemma (informally))

Two prisoners are interrogated separately, and have the options to either cooperate (C) with their fellow prisoner and stay silent, or defect (D) and accuse the fellow prisoner of the crime.

Possible outcomes:

- **Both cooperate:** no hard evidence against either of them, only short prison sentences for both.
- **One cooperates, the other defects:** the defecting prisoner is set free immediately, and the cooperating prisoner gets a very long prison sentence.
- **Both confess:** both get medium-length prison sentences.

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Example (Prisoner's Dilemma (payoff matrix))

Strategies $A_1 = A_2 = \{C, D\}$.

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3, 3	0, 4
	<i>D</i>	4, 0	1, 1

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An anti-coordination game:

Example (Hawk and Dove (informally))

In a fight for resources two players can behave either like a dove (D), yielding, or like a hawk (H), attacking.

Possible outcomes:

- **Both players behave like doves:** both players share the benefit.
- **A hawk meets a dove:** the hawk wins and gets the bigger part.
- **Both players behave like hawks:** the benefit gets lost completely because they will fight each other.

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Example (Hawk and Dove (payoff matrix))

Strategies $A_1 = A_2 = \{D, H\}$.

		player 2	
		<i>D</i>	<i>H</i>
player 1	<i>D</i>	3, 3	1, 4
	<i>H</i>	4, 1	0, 0

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A strictly competitive game:

Example (Matching Pennies (informally))

Two players can choose either heads (H) or tails (T) of a coin.

Possible outcomes:

- Both players make the same choice: player 1 receives one Euro from player 2.
- The players make different choices: player 2 receives one Euro from player 1.

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Example (Matching Pennies (payoff matrix))

Strategies $A_1 = A_2 = \{H, T\}$.

		player 2	
		<i>H</i>	<i>T</i>
player 1	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

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Bach or Stravinsky (aka Battle of the Sexes)



A coordination game:

Example (Bach or Stravinsky (informally))

Two persons, one of whom prefers Bach whereas the other prefers Stravinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite one. Let B be the action of going to the Bach concert and S the action of going to the Stravinsky concert.

Possible outcomes:

- **Both players make the same choice:** the player whose preferred option is chosen gets high payoff, the other player gets medium payoff.
- **The players make different choices:** they both get zero payoff.

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Bach or Stravinsky (aka Battle of the Sexes)



Example (Bach or Stravinsky (payoff matrix))

Strategies $A_1 = A_2 = \{B, S\}$.

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2, 1	0, 0
	<i>S</i>	0, 0	1, 2

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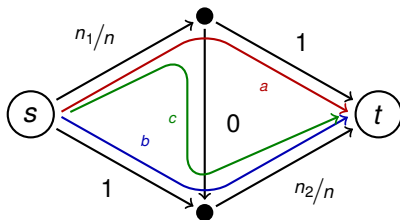
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Example (A congestion game)



player 2

		<i>a</i>	<i>b</i>	<i>c</i>
player 1	<i>a</i>	-2, -2	-1.5, -1.5	-2, -1.5
	<i>b</i>	-1.5, -1.5	-2, -2	-2, -1.5
	<i>c</i>	-1.5, -2	-1.5, -2	-2, -2

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2 Solution Concepts and Notation



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Question: What is a “solution” of a strategic game?

Answer:

- A strategy profile where all players play strategies that are **rational** (i. e., in some sense optimal).
- **Note:** There are different ways of making the above item precise (different solution concepts).
- A **solution concept** is a formal rule for predicting how a game will be played.

In the following, we will consider some solution concepts:

- Iterated dominance
- Nash equilibrium
- (Subgame-perfect equilibrium)

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Notation: we want to write down strategy profiles where one player's strategy is removed or replaced.

Let $a = (a_1, \dots, a_{|N|}) \in A = \prod_{i \in N} A_i$ be a strategy profile.

We write:

- $A_{-i} := \prod_{j \in N \setminus \{i\}} A_j$,
- $a_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{|N|})$, and
- $(a_{-i}, a'_i) := (a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_{|N|})$.

Example

Let $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $A_3 = \{X, Y, Z\}$, and $a := (T, R, Z)$.

Then $a_{-1} = (R, Z)$, $a_{-2} = (T, Z)$, $a_{-3} = (T, R)$.

Moreover, $(a_{-2}, L) = (T, L, Z)$.

3 Dominated Strategies



- Strictly Dominated Strategies
- Weakly Dominated Strategies

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Question: What strategy should an agent avoid?

One answer:

- **Eliminate** all obviously **irrational strategies**.
- A strategy is obviously **irrational** if there is **another strategy that is always better**, no matter what the other players do.

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Definition (Strictly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called **strictly dominated** in G if there is a strategy $a_i^+ \in A_i$ such that for all strategy profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ **strictly dominates** a_i .

If $a_i^+ \in A_i$ strictly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ **strictly dominant** in G .

Remark: Playing strictly dominated strategies is irrational.

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This suggests a solution concept:

iterative elimination of strictly dominated strategies:

while some strictly dominated strategy is left:

eliminate some strictly dominated strategy

if a unique strategy profile remains:

this unique profile is the solution

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Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3,3	0,4
	<i>D</i>	4,0	1,1

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Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3, 3	0, 4
	<i>D</i>	4, 0	1, 1

- **Step 1:** eliminate row *C* (strictly dominated by row *D*)

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Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		C	D
player 1	C	3, 3	0, 4
	D	4, 0	1, 1

- Step 1: eliminate row *C* (strictly dominated by row *D*)
- Step 2: eliminate column *C* (strictly dominated by col. *D*)

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Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		C	D
player 1	C	3, 3	0, 4
	D	4, 0	1, 1

- Step 1: eliminate row *C* (strictly dominated by row *D*)
- Step 2: eliminate column *C* (strictly dominated by col. *D*)

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Example (Iterative elim. of strictly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2,1	0,0
	<i>M</i>	1,2	2,1
	<i>B</i>	0,0	1,1

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Example (Iterative elim. of strictly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	1, 2	2, 1
	<i>B</i>	0, 0	1, 1

- **Step 1:** eliminate row *B* (strictly dominated by row *M*)

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Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	R
player 1	T	2, 1	0, 0
	M	1, 2	2, 1
	B	0, 0	1, 1

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)

Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	R
player 1	T	2, 1	0, 0
	M	1, 2	2, 1
	B	0, 0	1, 1

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)
- Step 3: eliminate row *M* (strictly dominated by row *T*)

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Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	R
player 1	T	2, 1	0, 0
	M	1, 2	2, 1
	B	0, 0	1, 1

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)
- Step 3: eliminate row *M* (strictly dominated by row *T*)

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Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2, 1	0, 0
	<i>S</i>	0, 0	1, 2

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Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2, 1	0, 0
	<i>S</i>	0, 0	1, 2

- No strictly dominated strategies.
- All strategies survive iterative elimination of strictly dominated strategies.
- All strategies **rationalizable**.

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Remark

Strict dominance between actions is rather rare.

We should identify more constraints on “solutions”, better solution concepts.

Proposition

The result of iterative elimination of strictly dominated strategies is unique, i. e., independent of the elimination order.

Proof.

Homework. □

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Definition (Weakly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called **weakly dominated** in G if there is a strategy $a_i^+ \in A_i$ such that for all profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+)$$

and that for at least one profile $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ **weakly dominates** a_i .

If $a_i^+ \in A_i$ weakly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ **weakly dominant** in G .

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What about
iterative elimination of weakly dominated strategies
as a solution concept?
Let's see what happens.

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Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	2, 1	1, 1
	<i>B</i>	0, 0	1, 1

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Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	2, 1	1, 1
	<i>B</i>	0, 0	1, 1

- **Step 1:** eliminate row *B* (weakly dominated by row *M*, $u_1(M, L) = 2 > 0 = u_1(B, L)$ and $u_1(M, R) = 1 = u_1(B, R)$)

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Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	R
player 1	T	2, 1	0, 0
	M	2, 1	1, 1
	B	0, 0	1, 1

- **Step 1:** eliminate row *B* (weakly dominated by row *M*, $u_1(M, L) = 2 > 0 = u_1(B, L)$ and $u_1(M, R) = 1 = u_1(B, R)$)
- **Step 2:** eliminate column *R* (weakly dominated by col. *L*)

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Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	R
player 1	T	2, 1	0, 0
	M	2, 1	1, 1
	B	0, 0	1, 1

- **Step 1:** eliminate row *B* (weakly dominated by row *M*, $u_1(M, L) = 2 > 0 = u_1(B, L)$ and $u_1(M, R) = 1 = u_1(B, R)$)
- **Step 2:** eliminate column *R* (weakly dominated by col. *L*)

Here, two solution profiles remain.

Iterative elimination of weakly dominated strategies:

- leads to **smaller games**,
- can also lead to situations where only a single solution remains,
- **but:** the result can depend on the elimination order!
(see example on next slide)

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Example (Iterative elim. of weakly dominated strategies)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	2, 1	1, 1
	<i>B</i>	0, 0	1, 1

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		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	0, 0
	<i>M</i>	2, 1	1, 1
	<i>B</i>	0, 0	1, 1

- **Step 1:** eliminate row *T* (weakly dominated by row *M*)

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Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	R
player 1	T	2, 1	0, 0
	M	2, 1	1, 1
	B	0, 0	1, 1

- **Step 1:** eliminate row *T* (weakly dominated by row *M*)
- **Step 2:** eliminate column *L* (weakly dominated by col. *R*)

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Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	R
player 1	T	2, 1	0, 0
	M	2, 1	1, 1
	B	0, 0	1, 1

- Step 1: eliminate row T (weakly dominated by row M)
- Step 2: eliminate column L (weakly dominated by col. R)

Different elimination order, different result,
even different payoffs (1, 1 vs. 2, 1)!

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- Definitions and Examples
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Question: Which strategy profiles are **stable**?

Possible answer:

- Strategy profiles where **no player benefits from playing a different strategy**
- **Equivalently**: Strategy profiles where every player's strategy is a **best response** to the other players' strategies

Such strategy profiles are called **Nash equilibria**, one of the **most-used solution concepts** in game theory.

Remark: In following examples, for non-Nash equilibria, only one possible profitable deviation is shown (even if there are more).

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Definition (Nash equilibrium)

A **Nash equilibrium** of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$,

$$u_i(a^*) \geq u_i(a_{-i}^*, a_j) \quad \text{for all } a_j \in A_j.$$

Remark: There is an alternative definition of Nash equilibria (which we consider because it gives us a slightly different perspective on Nash equilibria).

Definition (Best response)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game, $i \in N$ a player, and $a_{-i} \in A_{-i}$ a strategy profile of the players other than i .

Then a strategy $a_i \in A_i$ is a **best response** of player i to a_{-i} if

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a'_i) \quad \text{for all } a'_i \in A_i.$$

We write $B_i(a_{-i})$ for the set of best responses of player i to a_{-i} .

For a strategy profile $a \in A$, we write $B(a) = \prod_{i \in N} B_i(a_{-i})$.

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Definition (Nash equilibrium, alternative 1)

A **Nash equilibrium** of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$, $a_i^* \in B_i(a_{-i}^*)$.

Definition (Nash equilibrium, alternative 2)

A **Nash equilibrium** of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that $a^* \in B(a^*)$.

Proposition

The three definitions of Nash equilibria are equivalent.

Proof.

Homework. □

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Example (Nash Equilibria in the Prisoner's Dilemma)

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3, 3	0, 4
	<i>D</i>	4, 0	1, 1

- (C, C) : No Nash equilibrium (player 1: $C \rightarrow D$)
- (C, D) : No Nash equilibrium (player 1: $C \rightarrow D$)
- (D, C) : No Nash equilibrium (player 2: $C \rightarrow D$)
- (D, D) : Nash equilibrium!

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Example (Nash Equilibria in Hawk and Dove)

		player 2	
		<i>D</i>	<i>H</i>
player 1	<i>D</i>	3, 3	1, 4
	<i>H</i>	4, 1	0, 0

- (D, D) : No Nash equilibrium (player 1: $D \rightarrow H$)
- (D, H) : Nash equilibrium!
- (H, D) : Nash equilibrium!
- (H, H) : No Nash equilibrium (player 1: $H \rightarrow D$)

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Example (Nash Equilibria in Matching Pennies)

		player 2	
		<i>H</i>	<i>T</i>
player 1	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

- (H, H) : No Nash equilibrium (player 2: $H \rightarrow T$)
- (H, T) : No Nash equilibrium (player 1: $H \rightarrow T$)
- (T, H) : No Nash equilibrium (player 1: $T \rightarrow H$)
- (T, T) : No Nash equilibrium (player 2: $T \rightarrow H$)

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Example (Nash Equilibria in Bach or Stravinsky)

		Stravinsky enthusiast	
		<i>B</i>	<i>S</i>
Bach enthusiast	<i>B</i>	2,1	0,0
	<i>S</i>	0,0	1,2

- (B, B) : Nash equilibrium!
- (B, S) : No Nash equilibrium (player 1: $B \rightarrow S$)
- (S, B) : No Nash equilibrium (player 2: $S \rightarrow B$)
- (S, S) : Nash equilibrium!

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Example: Sealed-Bid Auctions



We consider a slightly larger example: **sealed-bid auctions**

Setting:

- An **object** has to be **assigned** to a winning bidder in exchange for a **payment**.
- For each player (“bidder”) $i = 1, \dots, n$, let v_i be the **private value** that bidder i assigns to the object.
(We assume that $v_1 > v_2 > \dots > v_n > 0$.)
- The bidders simultaneously give their **bids** $b_i \geq 0$, $i = 1, \dots, n$.
- The object is given to the bidder i with the **highest bid** b_i .
(Ties are broken in favor of bidders with lower index, i.e., if $b_i = b_j$ are the highest bids, then bidder i will win iff $i < j$.)

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Example: Sealed-Bid Auctions



Question: What should the winning bidder have to **pay**?

One possible answer: The highest bid.

Definition (First-price sealed-bid auction)

- $N = \{1, \dots, n\}$ with $v_1 > v_2 > \dots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ **wins** if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $$u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - b_i & \text{otherwise} \end{cases}$$
 where $b = (b_1, \dots, b_n)$.

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Example: Sealed-Bid Auctions



Example (First-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$\begin{array}{lll} v_1 = 100, & v_2 = 80, & v_3 = 53, \\ b_1 = 90, & b_2 = 85, & b_3 = 45. \end{array}$$

Observations:

- Bidder 1 wins, pays 90, gets utility $u_1(b) = v_1 - b_1 = 100 - 90 = 10$.
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding $b'_1 = 85$ instead. Then $u_1(b_{-1}, b'_1) = v_1 - b'_1 = 100 - 85 = 15$.

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Example: Sealed-Bid Auctions



Question: How to avoid **untruthful bidding** and **incentivize truthful revelation** of private valuations?

Different answer to question about payments: Winner pays the **second-highest** bid.

Definition (Second-price sealed-bid auction)

- $N = \{1, \dots, n\}$ with $v_1 > v_2 > \dots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ **wins** if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $$u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - \max_{j \neq i} b_j & \text{otherwise} \end{cases}$$
 where $b = (b_1, \dots, b_n)$.

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Example (Second-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$\begin{array}{lll} v_1 = 100, & v_2 = 80, & v_3 = 53, \\ b_1 = 90, & b_2 = 85, & b_3 = 45. \end{array}$$

Observations:

- Bidder 1 wins, pays 85, gets utility $u_1(b) = v_1 - b_2 = 100 - 85 = 15$.
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess the other bidders' private valuations.

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Example: Sealed-Bid Auctions



Proposition

In a second-price sealed-bid auction, bidding one's own valuation, $b_i^+ = v_i$, is a weakly dominant strategy.

Proof.

We have to show that b_i^+ weakly dominates **every** other strategy b_i of player i .

For that, it suffices to show that

1 for all $b_{-i} \in A_{-i}$, we have

$$u_i(b_{-i}, b_i^+) \geq u_i(b_{-i}, b_i) \text{ for all } b_{-i} \in A_{-i}, \text{ and that}$$

2 for all $b_{-i} \in A_{-i}$, we have

$$u_i(b_{-i}, b_i^+) > u_i(b_{-i}, b_i) \text{ for at least one } b_{-i} \in A_{-i}.$$

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Proof (ctd.)

Ad (1) [regardless of what the other bidders do,
 b_i^+ is always a best response]:

- Case I) bidder i wins:

bidder i pays $\max b_{-i} \leq v_i$, gets $u_i(b_{-i}, b_i^+) \geq 0$.

- Case I.a) bidder i decreases bid:

this does not help, since he might still win and pay the same as before, or lose and get utility 0.

- Case I.b) bidder i increases bid:

bidder i still wins and pays the same as before.

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Proof (ctd.)

Ad (1) (ctd.):

- **Case II) bidder i loses:**

bidder i pays nothing, gets $u_i(b_{-i}, b_i^+) = 0$.

- **Case II.a) bidder i decreases bid:**

bidder i still loses and gets utility 0.

- **Case II.b) bidder i increases bid:**

either bidder i still loses and gets utility 0, or becomes the winner and pays more than the object is worth to him, leading to a negative utility.

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Proof (ctd.)

Ad (2) [for each alternative b_i to b_i^+ , there is an opponent profile b_{-i} against which b_i^+ is strictly better than b_i]:

Let b_i be some strategy other than b_i^+ .

■ **Case I) $b_i < b_i^+$:**

Consider b_{-i} with $b_i < \max b_{-i} < b_i^+$.

With b_i , bidder i does not win any more, i. e., we have

$$u_i(b_{-i}, b_i^+) > 0 = u_i(b_{-i}, b_i).$$

■ **Case II) $b_i > b_i^+$:**

Consider b_{-i} with $b_i > \max b_{-i} > b_i^+$.

With b_i , bidder i overbids and pays more than the object is

worth to him, i. e., we have $u_i(b_{-i}, b_i^+) = 0 > u_i(b_{-i}, b_i)$.

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Example: Sealed-Bid Auctions



Proposition

Profiles of weakly dominant strategies are Nash equilibria.

Proof.

Homework. □

Proposition

In a second-price sealed-bid auction, if all bidders bid their true valuations, this is a Nash equilibrium.

Proof.

Follows immediately from the previous two propositions. □

Remark: This is not the only Nash equilibrium in second-price sealed-bid auctions, though.

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Motivation: We have seen **two different solution concepts**,

- Surviving iterative elimination of (strictly) **dominated strategies** and
- **Nash equilibria**.

Obvious question: Is there any **relationship** between the two?

Answer: Yes, Nash equilibria refine the concept of iterative elimination of strictly dominated strategies. We will formalize this on the next slides.

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Lemma (preservation of Nash equilibria)

Let G and G' be two strategic games where G' is obtained from G by elimination of one strictly dominated strategy.

Then a strategy profile a^ is a Nash equilibrium of G if and only if it is Nash equilibrium of G' .*

Proof.

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ and $G' = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$.

Let a'_i be the eliminated strategy.

Then there is a strategy a_i^+ such that for all $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a'_i) < u_i(a_{-i}, a_i^+). \quad (1)$$

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Proof (ctd.)

“ \Rightarrow ”: Let a^* be a Nash equilibrium of G .

- **Nash equilibrium strategies are not eliminated:** For players $j \neq i$, this is clear, because none of their strategies are eliminated.

For player i , action a_i^* is a best response to a_{-i}^* , and in particular at least as good a response as a_i^+ :

$$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

With (1) $u_i(a_{-i}, a_i^+) > u_i(a_{-i}, a_i')$, we get $u_i(a_{-i}^*, a_i^+) > u_i(a_{-i}^*, a_i')$ and hence $a_i^* \neq a_i'$.

Thus, the Nash equilibrium strategy a_i^* is not eliminated.

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Proof (ctd.)

“ \Rightarrow ” (ctd.):

- **Best responses remain best responses:** For all players $j \in N$, a_j^* is a best response to a_{-j}^* in G . Since in G' , no potentially better responses are introduced ($A'_j \subseteq A_j$) and the payoffs are unchanged, this also holds in G' .

Hence, a^* is also a Nash equilibrium of G' .

“ \Leftarrow ”: Let a^* be a Nash equilibrium of G' .

- **For player $j \neq i$:** a_j^* is a best response to a_{-j}^* in G as well, since the responses available to player j in G and G' are the same.

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Proof (ctd.)

“ \Leftarrow ” (ctd.):

- For player i : Since $A_i = A'_i \cup \{a_i\}$ and a_i^* is a best response to a_{-i}^* among the strategies in A'_i , it suffices to show that a_i is no better response.

Because a^* is a Nash equilibrium in G' and a_i^+ is a strategy in A'_i , we have $u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+)$.

Since a_i^+ strictly dominates a_i , we have $u_i(a_{-i}^*, a_i^+) > u_i(a_{-i}^*, a_i)$, and hence $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i)$.

Therefore, a_i cannot be a better response to a_{-i}^* than a_i^* .

Hence, a^* is also a Nash equilibrium of G . □

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Corollary

If iterative elimination of strictly dominated strategies results in a *unique* strategy profile a^* , then a^* is the unique Nash equilibrium of the original game.

Proof.

Assume that a^* is the unique remaining strategy profile. By definition, a^* must be a Nash equilibrium of the remaining game.

We can inductively apply the previous lemma (preservation of Nash equilibria) and see that a^* (and no other strategy profile) must have been a Nash equilibrium before the last elimination step, and before that step, \dots , and in the original game. \square

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Playing it Safe (in Two-Player Games)



Motivation: What happens if both players try to “play it safe”?

Question: What does it even mean to “play it safe”?

Answer: Choose a strategy that guarantees the **highest worst-case payoff**.

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Playing it Safe (in Two-Player Games)

Example

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 1	2, -20
	<i>M</i>	3, 0	-10, 1
	<i>B</i>	-100, 2	3, 3

Worst-case payoff for **player 1**:

- if playing *T*: 2
- if playing *M*: -10
- if playing *B*: -100

↪ play *T*.

Worst-case payoff for **player 2**:

- if playing *L*: 0
- if playing *R*: -20

↪ play *L*.

However: Unlike (B, R) , the profile (T, L) is **not** a Nash equilibrium.

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Playing it Safe (in Two-Player Games)

Example

		player 2	
		L	R
player 1	T	2, 1	2, -20
	M	3, 0	-10, 1
	B	-100, 2	3, 3

Worst-case payoff for player 1:

- if playing T : 2
- if playing M : -10
- if playing B : -100

↪ play T .

Worst-case payoff for player 2:

- if playing L : 0
- if playing R : -20

↪ play L .

However: Unlike (B, R) , the profile (T, L) is **not** a Nash equilibrium.

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Playing it Safe (in Two-Player Games)

Example

		player 2	
		L	R
player 1	T	2, 1	2, -20
	M	3, 0	-10, 1
	B	-100, 2	3, 3

Worst-case payoff for player 1:

- if playing T : 2
- if playing M : -10
- if playing B : -100

↪ play T .

Worst-case payoff for player 2:

- if playing L : 0
- if playing R : -20

↪ play L .

However: Unlike (B, R) , the profile (T, L) is **not** a Nash equilibrium.

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Playing it Safe (in Two-Player Games)



Observation: In general, pairs of **maximinimizers**, like (T, L) in the example above, are **not** the same as Nash equilibria.

Claim: However, in **zero-sum games**, pairs of maximinimizers and Nash equilibria **are essentially the same**.

(Tiny restriction: This does not hold if the considered game has no Nash equilibrium at all, because unlike Nash equilibria, pairs of maximinimizers always exist.)

Reason (intuitively): In **zero-sum games**, the **worst-case assumption** that the other player tries to harm you as much as possible is **justified**, because harming the other is the same as maximizing ones own payoff. **Playing it safe is rational**.

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Definition (Zero-sum game)

A **zero-sum game** is a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and

$$u_1(a) = -u_2(a)$$

for all $a \in A$.

Example (Matching Pennies as a zero-sum game)

		player 2	
		<i>H</i>	<i>T</i>
player 1	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

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Definition (Maximinimizer)

Let $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game.

An action $x^* \in A_1$ is called **maximinimizer** for player 1 in G if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \quad \text{for all } x \in A_1,$$

and $y^* \in A_2$ is called **maximinimizer** for player 2 in G if

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \quad \text{for all } y \in A_2.$$

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Example (Zero-sum game with three actions each)

		player 2		
		<i>L</i>	<i>C</i>	<i>R</i>
player 1	<i>T</i>	8, -8	3, -3	-6, 6
	<i>M</i>	2, -2	-1, 1	3, -3
	<i>B</i>	-6, 6	4, -4	8, -8

Guaranteed worst-case payoffs:

- $T: -6, M: -1, B: -6 \rightsquigarrow$ maximinimizer M
- $L: -8, C: -4, R: -8 \rightsquigarrow$ maximinimizer C

\rightsquigarrow pair of maximinimizers (M, C) with payoffs $(-1, 1)$
(not a Nash equilibrium; this game has no Nash equilibrium.)

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Example (Maximinimization vs. minimaximization)

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	1, -1	2, -2
	<i>B</i>	-2, 2	-4, 4

Worst-case payoffs (player 2):

- $L: -1, R: -2$
- Maximize: -1

Best-case payoffs (player 1):

- $L: +1, R: +2$
- Minimize: $+1$

Observation: Results identical up to different sign.

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Lemma

Let $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = - \min_{y \in A_2} \max_{x \in A_1} u_1(x, y). \quad (2)$$

Proof.

For any real-valued function f , we have

$$\min_z -f(z) = - \max_z f(z). \quad (3)$$

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Proof (ctd.)

Thus, for all $y \in A_2$,

$$\begin{aligned} - \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) &\stackrel{(3)}{=} \max_{y \in A_2} - \max_{x \in A_1} u_1(x, y) \\ &\stackrel{(3)}{=} \max_{y \in A_2} \min_{x \in A_1} -u_1(x, y) \\ &\stackrel{ZS}{=} \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \end{aligned}$$

□

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Now, we are ready to prove our
main theorem about zero-sum games and Nash equilibria.

In zero-sum games:

- 1 Every Nash equilibrium is a pair of maximinimizers.
- 2 All Nash equilibria have the same payoffs.
- 3 If there is at least one Nash equilibrium, then every pair of maximinimizers is a Nash equilibrium.

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Theorem (Maximinimizer theorem)

Let $G = (\{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a zero-sum game. Then:

- 1 If (x^*, y^*) is a Nash equilibrium of G , then x^* and y^* are maximinimizers for player 1 and player 2, respectively.
- 2 If (x^*, y^*) is a Nash equilibrium of G , then

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) = u_1(x^*, y^*).$$

- 3 If $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$, and x^* and y^* maximinimizers of player 1 and player 2 respectively, then (x^*, y^*) is a Nash equilibrium.

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Proof.

1 Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this implies

$$u_1(x^*, y^*) \leq u_1(x^*, y) \quad \text{for all } y \in A_2.$$

Thus

$$u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (4)$$

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Proof (ctd.)

1 (ctd.)

Furthermore, since (x^*, y^*) is a Nash equilibrium, also

$$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

Hence

$$u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).$$

This implies

$$u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (5)$$

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Proof (ctd.)

1 (ctd.)

Inequalities (4) and (5) together imply that

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad (6)$$

Thus, x^* is a maximinimizer for player 1.

Similarly, we can show that y^* is a maximinimizer for player 2:

$$u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \quad (7)$$

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Proof (ctd.)

2 We only need to put things together:

$$\begin{aligned} \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) &\stackrel{(6)}{=} u_1(x^*, y^*) \\ &\stackrel{\text{ZS}}{=} -u_2(x^*, y^*) \\ &\stackrel{(7)}{=} -\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) \\ &\stackrel{(2)}{=} \min_{y \in A_2} \max_{x \in A_1} u_1(x, y). \end{aligned}$$

In particular, it follows that all Nash equilibria share the same payoff profile.

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Proof (ctd.)

- 3 Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) =: v^*. \quad (8)$$

With Equation (2) from the previous lemma, we get

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = -v^*. \quad (9)$$

With x^* and y^* being maximinimizers, (8) and (9) imply

$$u_1(x^*, y) \geq v^* \quad \text{for all } y \in A_2, \text{ and} \quad (10)$$

$$u_2(x, y^*) \geq -v^* \quad \text{for all } x \in A_1. \quad (11)$$

Proof (ctd.)

3 (ctd.)

Special cases of (10) and (11) for $x = x^*$ and $y = y^*$:

$$u_1(x^*, y^*) \geq v^* \quad \text{and} \quad u_2(x^*, y^*) \geq -v^*.$$

With $u_1 = -u_2$, the latter is equivalent to $u_1(x^*, y^*) \leq v^*$, which gives us

$$u_1(x^*, y^*) = v^*. \quad (12)$$

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Proof (ctd.)

3 (ctd.)

Plugging (12) into the right-hand side of (10) gives us

$$u_1(x^*, y) \geq u_1(x^*, y^*) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this is equivalent to

$$u_2(x^*, y) \leq u_2(x^*, y^*) \quad \text{for all } y \in A_2.$$

In other words, y^* is a best response to x^* .

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Proof (ctd.)

3 (ctd.)

Similarly, we can plug (12) into the right-hand side of (11) and obtain

$$u_2(x, y^*) \geq -u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

Again using $u_1 = -u_2$, this is equivalent to

$$u_1(x, y^*) \leq u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

In words, x^* is also a best response to y^* .

Hence, (x^*, y^*) is a Nash equilibrium.



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Corollary

Let $G = \langle \{1, 2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game, and let (x_1^*, y_1^*) and (x_2^*, y_2^*) be two Nash equilibria of G .

Then (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G .

In other words: Nash equilibria of zero-sum games can be arbitrarily recombined.

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Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

With part (2) of the maximinimizer theorem, we get that $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$.

With this equality, with x_1^* , x_2^* , y_1^* , and y_2^* all being maximinimizers, and with part (3) of the maximinimizer theorem, we get that (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G . □

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Summary



- **Strategic games** are one-shot games of finitely many players with given action sets and payoff functions. Players have perfect information.
- **Solution concepts:** survival of **iterative elimination of strictly dominated strategies**, **Nash equilibria**.
- **Relation between solution concepts:** Nash equilibria always survive iterative elimination of strictly dominated strategies.
- In **zero-sum games**, one player's gain is the other player's loss. Thus, playing it safe is rational. Relevant concept: **maximinimizers**.
- **Relation to Nash equilibria:** In zero-sum games, Nash equilibria are pairs of maximinimizers, and, if at least one Nash equilibrium exists, pairs of maximinimizers are also Nash equilibria.

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