Dynamic Epistemic Logic

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Exercise Sheet 3

Due: May 15th, 2019, 16:00

Exercise 3.1 (**S5**: Axioms and frame properties I, 6 points)

A Kripke frame $\mathcal{F} = \langle S, R\rangle$ is defined exactly like a Kripke model $\langle S, R, V\rangle$, but without the valuation V. The set of all models over $\langle S, R\rangle$ is the set of all models $\langle S, R, V\rangle$ where V is any propositional valuation. A formula is valid in a frame $\mathcal{F}$, if it is valid in all models over $\mathcal{F}$. It is valid in a class of frames, if it is valid in each frame in that class. We say that an axiom defines a class of frames if the axiom is valid exactly in this class of frames. Show that

(a) the axiom **T** defines the class of *reflexive* frames,

(b) the axiom **4** defines the class of *transitive* frames,

(c) the axiom **5** defines the class of *Euclidean* frames.

Note: You might be able to re-use parts of your solutions for Exercise 2.3.

Exercise 3.2 (n-bisimulation, 4 points)

Let two models $\mathcal{M} = \langle S, R, V\rangle$ and $\mathcal{M}' = \langle S', R', V'\rangle$ be given. For any natural number n, we define two states $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ to be n-bisimilar, writing $(\mathcal{M}, s) \leftrightarroweq_n (\mathcal{M}', s')$, iff

(atoms) $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$,

(forth) $n = 0$ or (if $n > 0$) for all $a \in A$ and $t \in S$ such that $(s, t) \in R_a$, there is also a $t' \in S'$ such that $(s', t') \in R'$ and $(\mathcal{M}, t) \leftrightarroweq_{n-1} (\mathcal{M}', t')$, and

(back) $n = 0$ or (if $n > 0$) for all $a \in A$ and $t' \in S'$ such that $(s', t') \in R'_a$, there is also a $t \in S$ such that $(s, t) \in R$ and $(\mathcal{M}, t) \leftrightarroweq_{n-1} (\mathcal{M}', t')$.

Furthermore, we define the *modal depth* of $\mathcal{L}_K$-formulas as

$$\begin{aligned} depth(p) &= 0 \text{ if } p \text{ is an atomic proposition} \\ depth(\neg\phi) &= depth(\phi) \\ depth(\phi \wedge \psi) &= \max\{depth(\phi), depth(\psi)\} \\ depth(K_a\phi) &= 1 + depth(\phi). \end{aligned}$$

We say that the states $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are epistemically equivalent up to depth $n \in \mathbb{N}$ and write $(\mathcal{M}, s) \equiv^n_{\mathcal{L}_K} (\mathcal{M}', s')$ if and only if $\mathcal{M}, s \models \phi$ iff $\mathcal{M}', s' \models \phi$ for all formulas $\phi \in \mathcal{L}_K$ with $depth(\phi) \leq n$. Show that $(\mathcal{M}, s) \equiv^n_{\mathcal{L}_K} (\mathcal{M}', s')$ if and only if $(\mathcal{M}, s) \leftrightarroweq_n (\mathcal{M}', s')$.

Exercise 3.3 (**S5**: Deriving theorems, 1+1 points)

Derive the following **S5** theorems. Recall that a derivation is a finite sequence of formulas, such that each formula is either an instance of one of the axioms, an instance of a propositional tautology, or the result of the application of one of the rules (necessitation, modus ponens) on previous formulas.

(a) $K_a(p \to p)$

(b) $C_B p \leftrightarrow C_B C_B p$