Dynamic Epistemic Logic

4. Action Models

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Introduction

Action Models

So far: Only public announcements.

Now: How to model other ways of knowledge changes, such as private announcements, sensing, or ontic (world-changing) actions that affect knowledge along the way?

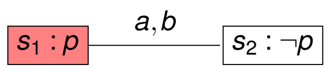
Idea: Action models similar to epistemic models.

Action Models

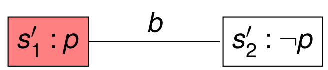
Example

Agents *a* and *b* both don't know the value of proposition *p*. This is common knowledge among them. In fact, *p* is true. Then agent *a* receives a letter containing the value of *p* and reads it. Agent *b* observes *a* reading the letter and knows that it is about *p*, but *b* does not learn the value of *p*.

Model Before:



Model After:

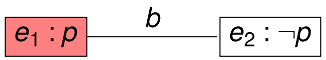


Question: How to get from Before to After?

Answer: Action models.

Remark: After, $s_1' \models$
$K_a p \wedge (\neg K_b p \wedge \neg K_b \neg p) \wedge K_b(K_a p \vee K_a \neg p) \wedge K_a(\neg K_b p \wedge \neg K_b \neg p)$
$\rightsquigarrow$ action model needs to achieve exactly that!

Action model Read:

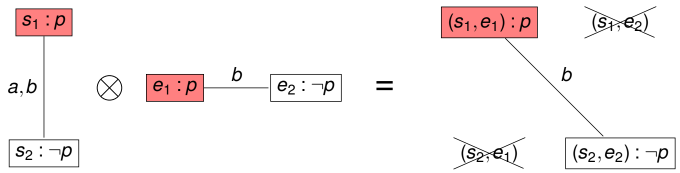


With this action model, After = Before $\otimes$ Read, for an appropriate definition of $\otimes$.

Definition (Product update, informally)

The product update $\otimes$ denotes a restricted modal update with component worlds (s, e) only present if $(\mathcal{M}, s) \models pre(e)$.

Model Before $\otimes$ Read:



- $(s_1, e_1) \sim_b (s_2, e_2)$ because $s_1 \sim_b s_2$ and $e_1 \sim_b e_2$.
- (s_1, e_2) and (s_2, e_1) were eliminated because e_2 cannot be applied in s_1 and e_1 cannot be applied in s_2.

Action models

Definition (Action model)

Let $\mathcal{L}$ be any logical language for a set of agents A and a set of atoms P. Then an $\mathcal{S}5$ action model M is a structure $(E, \sim, pre)$ such that:

- E is the domain of events,
- $\sim_a$ is an equivalence relation on E for all $a \in A$, the indistinguishability relation for agent a, and
- $pre : E \to \mathcal{L}$ is the precondition function that assigns a precondition $pre(e) \in \mathcal{L}$ to all $e \in E$.

A pointed action model is such a structure (M, e) with $e \in E$.

Syntax of Action Model Logic

Action Models

Definition (Language $\mathcal{L}_{KC\otimes}$)

Let P be a countable set of atomic propositions and A a finite set of agent symbols. Then the language $\mathcal{L}_{KC\otimes}$ of action model logic is the union of the formulas $\varphi \in \mathcal{L}^{\text{stat}}_{KC\otimes}$ and the actions $\alpha \in \mathcal{L}^{\text{act}}_{KC\otimes}$ defined by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid C_B\varphi \mid [\alpha]\varphi$$
$$\alpha ::= (M,e) \mid \alpha \cup \alpha$$

where $p \in P$, $a \in A$, $B \subseteq A$, and (M,e) is a pointed action model with a finite domain E, and

- for all events $e' \in E$, the precondition $pre(e')$ is a $\mathcal{L}^{\text{stat}}_{KC\otimes}$ formula that has already been constructed in a previous step of the induction.

Action Models

Intuition:

- $[\alpha]\varphi$: After (every) application of action α, φ is true.

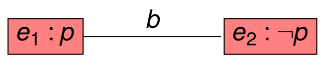
Abbreviations:

- $\langle\alpha\rangle\varphi := \neg[\alpha]\neg\varphi$
  After (some) application of action α, φ is true.
- $M := \bigcup_{e \in E}(M,e)$

Action Models

Deterministic vs. nondeterministic actions:

- $\alpha = (M,e)$: Deterministic action α with unique pointed event e. Example: $\alpha = (\text{Read}, e_1)$.
- $\alpha = \alpha_1 \cup \alpha_2$: Nondeterministic choice, i. e., either α_1 or α_2 happens. Example: $\alpha = (\text{Read}, e_1) \cup (\text{Read}, e_2) = \text{Read}$.
  - Remark 1a: $\alpha = \text{Read}$ not properly nondeterministic, since preconditions of e_1 and e_2 are mutually exclusive.
  - Remark 1b: We will see a properly nondeterministic action later (action Mayread).
  - Remark 2a: If, for $\alpha = (M_1, e_1) \cup (M_2, e_2)$, we have $M_1 = M_2$, then we can depict α as a multi-pointed model, like $(\text{Read}, e_1) \cup (\text{Read}, e_2)$:



  - Remark 2b: Formal introduction of multi-pointed models: later.

Example (Action model Read, formally)

Read is the action model $(\{e_1, e_2\}, \sim, pre)$ with

$$\sim_a = \{(e_1,e_1),(e_2,e_2)\} \qquad pre(e_1) = p$$
$$\sim_b = \{(e_1,e_1),(e_1,e_2),(e_2,e_1),(e_2,e_2)\} \qquad pre(e_2) = \neg p.$$

(and with pointed event e_1).

Remark: Public announcements are a special case of action models.

Example (Public announcements)

Action model for the public announcement of φ:

$$e : \varphi$$

Fix agents A and atomic propositions P.

Example (Skip)

Action skip (or **1**) is the pointed action model $((\{e\}, \sim, pre), e)$ with $pre(e) = \top$ and $\sim_a = \{(e,e)\}$ for all $a \in A$.

Example (Crash)

Action crash (or **0**) is the pointed action model $((\{e\}, \sim, pre), e)$ with $pre(e) = \bot$ and $\sim_a = \{(e,e)\}$ for all $a \in A$.

Question: Can we "chain" actions one after the other?

Definition (Composition)

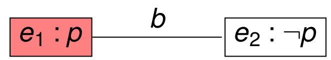
Let $M = (E, \sim, pre)$ and $M' = (E', \sim', pre')$ be action models in $\mathcal{L}^{act}_{KC\otimes}$. Then their composition $(M; M')$ is the action model $M'' = (E'', \sim'', pre'')$ such that:

- $E'' = E \times E'$,
- $(e,e') \sim''_a (\varepsilon, \varepsilon')$ iff $e \sim_a \varepsilon$ and $e' \sim'_a \varepsilon'$, and
- $pre''((e,e')) = \langle M, e\rangle pre'(e')$.

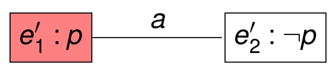
For pointed action models: $((M,e);(M',e')) = ((M;M'),(e,e'))$.

Example (Composition)

Action model $(\text{Read}_a, e_1) = (\text{Read}, e_1)$:



Action model (Read_b, e'_1):

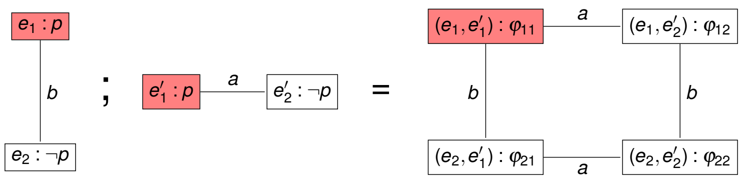


[...]

Action Models

Example (Composition, ctd.)

Action model $(\text{Read}_a, e_1); (\text{Read}_b, e_1')$:



where

$$\varphi_{11} = \langle \text{Read}_a, e_1 \rangle p \equiv p \qquad \varphi_{12} = \langle \text{Read}_a, e_1 \rangle \neg p \equiv \bot$$
$$\varphi_{21} = \langle \text{Read}_a, e_2 \rangle p \equiv \bot \qquad \varphi_{22} = \langle \text{Read}_a, e_2 \rangle \neg p \equiv \neg p$$

Action Models

Example (Composition, ctd.)

Remark: With $\varphi_{12} \equiv \varphi_{21} \equiv \bot$, events (e_1, e_2') and (e_1', e_2) can be eliminated as globally inapplicable.

This leaves us with $(\text{Read}_a, e_1); (\text{Read}_b, e_1')$ equivalent to:

$(e_1, e_1') : p$ $\qquad$ $(e_2, e_2') : \neg p$

Further eliminating unreachable events, we get:

$(e_1, e_1') : p$

In other words, if both a and b read the message that p is true, and they are aware of each other reading the message, the two actions combined must produce common knowledge of p.

Semantics of Action Model Logic

Action Models

Definition (Product update)

Let $\mathcal{M} = (S, \sim, V)$ be an epistemic model and let $M = (E, \sim, pre)$ be an action model. Then the product update $\mathcal{M} \otimes M$ is the epistemic model $\mathcal{M}' = (S', \sim', V')$ with:

- $S' = \{(s,e) \in S \times E \mid \mathcal{M}, s \models pre(e)\}$,
- $(s,e) \sim_a' (t,\varepsilon)$ iff $s \sim_a t$ and $e \sim_a \varepsilon$, for $a \in A$, and
- $(s,e) \in V_p'$ iff $s \in V_p$.

Example

$(\text{Before}, s_1) \otimes (\text{Read}, e_1) = (\text{After}, (s_1, e_1))$

Action Models

Definition (Semantics of formulas and actions)

Let $(\mathcal{M}, s)$ be an epistemic state, $\varphi \in \mathcal{L}^{stat}_{KC\otimes}$ and $\alpha \in \mathcal{L}^{act}_{KC\otimes}$.

$\mathcal{M}, s \models p,\ \neg\varphi,\ \varphi \wedge \psi,\ K_a\varphi,\ C_B\varphi$ as usual

$\mathcal{M}, s \models [\alpha]\varphi$ iff for all $(\mathcal{M}', s')$:
$(\mathcal{M}, s)[\![\alpha]\!](\mathcal{M}', s')$ implies $(\mathcal{M}', s') \models \varphi$

where

- $(\mathcal{M}, s)[\![(M, e)]\!](\mathcal{M}', s')$ iff
  $(\mathcal{M}, s) \models pre(e)$ and $(\mathcal{M}', s') = (\mathcal{M} \otimes M, (s, e))$, and
- $[\![\alpha \cup \alpha']\!] = [\![\alpha]\!] \cup [\![\alpha']\!]$.

Action Models

Remarks:

- For $\alpha = (M, e)$, $[\![\alpha]\!]$ is functional, i. e., for each $(\mathcal{M}, s)$, there is at most one $(\mathcal{M}', s')$ with $(\mathcal{M}, s)[\![(M, e)]\!](\mathcal{M}', s')$.
- For $\alpha = \alpha_1 \cup \alpha_2$, this is no longer necessarily the case. Careful with duality between $[\alpha]$ and $\langle\alpha\rangle$, then.

Special case $\alpha = (M, e)$: Then $\mathcal{M}, s \models [\alpha]\varphi$ iff $\mathcal{M}, s \models pre(e)$ implies $(\mathcal{M} \otimes M, (s, e)) \models \varphi$.

Dual $\langle\alpha\rangle$, for $\alpha = (M, e)$:

$\mathcal{M}, s \models \langle\alpha\rangle\varphi$ iff
$\mathcal{M}, s \not\models [\alpha]\neg\varphi$ iff
$\mathcal{M}, s \models pre(e)$ does not imply $(\mathcal{M} \otimes M, (s, e)) \models \neg\varphi$ iff
$\mathcal{M}, s \models pre(e)$ and $(\mathcal{M} \otimes M, (s, e)) \not\models \neg\varphi$ iff
$\mathcal{M}, s \models pre(e)$ and $(\mathcal{M} \otimes M, (s, e)) \models \varphi$

Action Models

Remark: This is very similar to the semantics of $[\varphi]\psi$ and $\langle\varphi\rangle\psi$ in public announcement logic.

For completeness, dual $\langle\alpha\rangle$, for general α:

$\mathcal{M}, s \models \langle\alpha\rangle\varphi$ iff
$\mathcal{M}, s \not\models [\alpha]\neg\varphi$ iff
not f. a. $(\mathcal{M}', s') : (\mathcal{M}, s)[\![\alpha]\!](\mathcal{M}', s')$ implies $(\mathcal{M}', s') \models \neg\varphi$ iff
there ex. $(\mathcal{M}', s') : (\mathcal{M}, s)[\![\alpha]\!](\mathcal{M}', s')$ and $(\mathcal{M}', s') \not\models \neg\varphi$ iff
there ex. $(\mathcal{M}', s') : (\mathcal{M}, s)[\![\alpha]\!](\mathcal{M}', s')$ and $(\mathcal{M}', s') \models \varphi$

Action Models

Proposition

Let $(M, e), (M', e') \in \mathcal{L}^{act}_{KC\otimes}$ and $\varphi \in \mathcal{L}^{stat}_{KC\otimes}$. Then $[(M, e); (M', e')]\varphi$ is equivalent to $[(M, e)][(M', e')]\varphi$.

Proof.

Let $(\mathcal{M}, s)$ be arbitrary. Show that $\mathcal{M}, s \models [(M, e); (M', e')]\varphi$ iff $\mathcal{M}, s \models [(M, e)][(M', e')]\varphi$. For this, it is sufficient to show that $\mathcal{M} \otimes (M; M')$ is isomorphic to $(\mathcal{M} \otimes M) \otimes M'$.

- Isomoporphic domains: Let $(s, (e, e')) \in \mathcal{D}(\mathcal{M} \otimes (M; M'))$. Then: $\mathcal{M}, s \models pre''((e, e')) = \langle M, e\rangle pre'(e')$ iff $\mathcal{M}, s \models pre(e) \wedge [M, e]pre'(e')$ iff $\mathcal{M}, s \models pre(e)$ (1) and $\mathcal{M}, s \models [M, e]pre'(e')$ (2). [...]

Action Models

Proof (ctd.)

- Isomoporphic domains (ctd.): [. . .] We have: $\mathcal{M}, s \models pre(e)$ (1) and $\mathcal{M}, s \models [M,e]pre'(e')$ (2). From (1): $(s,e) \in \mathcal{D}(\mathcal{M} \otimes M)$ (3). From (2) and (3): $(\mathcal{M} \otimes M, (s,e)) \models pre'(e')$. This implies $((s,e),e') \in \mathcal{D}((\mathcal{M} \otimes M) \otimes M')$. Conversely, we also get $(s,(e,e')) \in \mathcal{D}(\mathcal{M} \otimes (M,M'))$ for all $((s,e),e') \in \mathcal{D}((\mathcal{M} \otimes M) \otimes M')$.
- Accessibility relations: Assume that $(s,(e,e')) \sim_a (t,(\varepsilon,\varepsilon'))$. This holds iff $s \sim_a t$ and $(e,e') \sim_a (\varepsilon,\varepsilon')$ iff $s \sim_a t$ and $e \sim_a \varepsilon$ and $e' \sim_a \varepsilon'$ iff $(s,e) \sim_a (t,\varepsilon)$ and $e' \sim_a \varepsilon'$ iff $((s,e),e') \sim_a ((t,\varepsilon),\varepsilon')$.
- Valuations: clear. □

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The previous proposition states that composition "does the right thing", but only for composition of deterministic actions.

Question: What about composition of nondeterministic α?

Answer: No need to worry (cf. following two propositions).

Proposition

Let $\alpha, \beta, \gamma \in \mathcal{L}^{act}_{KC\otimes}$. Then

- $((\alpha \cup \beta);\gamma)$ *is equivalent to* $(\alpha;\gamma) \cup (\beta;\gamma)$, *and*
- $(\alpha;(\beta \cup \gamma))$ *is equivalent to* $(\alpha;\beta) \cup (\alpha;\gamma)$. □

Proposition

Let $\alpha, \beta \in \mathcal{L}^{act}_{KC\otimes}$ and $\varphi \in \mathcal{L}^{stat}_{KC\otimes}$. Then $[\alpha \cup \beta]\varphi$ is equivalent to $[\alpha]\varphi \wedge [\beta]\varphi$. □

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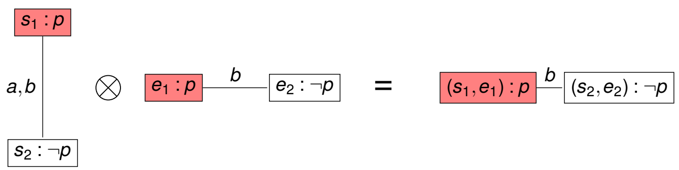
Example

Model $(\text{Before}, s_1) \otimes (\text{Read}, e_1)$:



Then:

- $\text{Before}, s_1 \models [\text{Read}, e_1]K_a p$
- $\text{Before}, s_1 \models [\text{Read}, e_1]\neg K_b K_a p$
- $\text{Before}, s_1 \models [\text{Read}, e_1]C_{ab}(K_a p \vee K_a \neg p)$

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Example

Now, *a* may only read the letter, but does not have to. Agent *b* does not know whether *a* will read it or not. Actually, *a* does not read the letter.

From *b*'s perspective, there are three possibilities:

- *a* reads the letter and learns that *p* is true.
- *a* reads the letter and learns that *p* is false.
- *a* does not read the letter and learns nothing about *p*.

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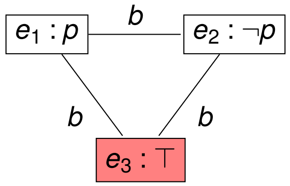
Example (ctd.)

Action model (Mayread, e_3):



$$\text{Mayread} = (\text{Mayread}, e_1) \cup (\text{Mayread}, e_2) \cup (\text{Mayread}, e_3)$$

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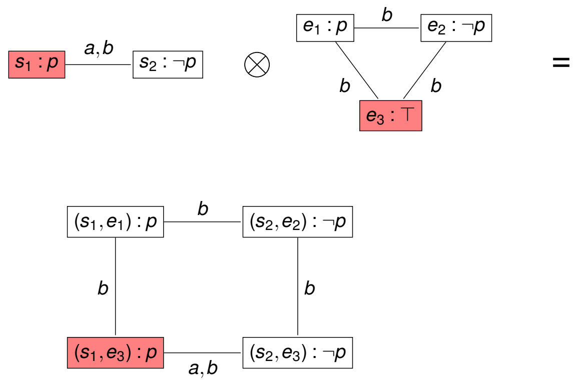
Example (ctd.)

Model (Before, s_1) $\otimes$ (Mayread, e_3):



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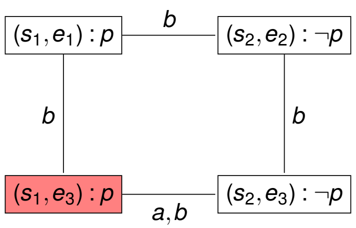
Example (ctd.)

Model (Before, s_1) $\otimes$ (Mayread, e_3):



- Before, $s_1 \models [\text{Mayread}, e_3]\neg(K_a p \vee K_a \neg p) \wedge \hat{K}_b(K_a p \vee K_a \neg p)$
- Before $\models p \rightarrow$ $(\langle\text{Mayread}\rangle K_a p \wedge \langle\text{Mayread}\rangle \neg K_a p \wedge \neg\langle\text{Mayread}\rangle K_a \neg p)$

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Bisimilarity and Action Emulation

Bisimilarity and Action Emulation

- Can two action models be bisimilar? $\rightsquigarrow$ Yes.
- Does the application of bisimilar action models to bisimilar epistemic states lead to bisimilar successor states? $\rightsquigarrow$ Yes.
- Do we even need bisimilarity of actions models for that? $\rightsquigarrow$ No.
- Weaker notion of emulation is enough.

Bisimilarity and Action Emulation

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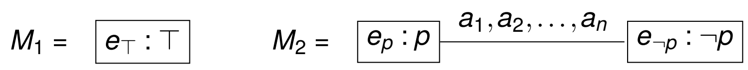
Example

M_1 and M_2 are not bisimilar, but always behave in the same way $\rightsquigarrow$ similar enough.



Before looking at bisimulations and emulations between action models, let us quickly see that applying the same action to two bisimilar epistemic states always results in bisimilar successor states.

Bisimilarity and Action Emulation

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Proposition (Preservation of bisimilarity)

Let $(\mathcal{M},s)$ *and* $(\mathcal{M}',s')$ *be two epistemic states with* $(\mathcal{M},s) \leftrightarrows (\mathcal{M}',s')$. *Let* (M,e) *with* $M = (E,\sim,pre)$ *be applicable in* $(\mathcal{M},s)$. *Then* $(\mathcal{M}\otimes M,(s,e)) \leftrightarrows (\mathcal{M}'\otimes M,(s',e))$.

Proof.

(M,e) is also applicable in $(\mathcal{M}',s')$, since $\mathcal{M},s \models pre(e)$ and $(\mathcal{M},s) \leftrightarrows (\mathcal{M}',s')$ implies $(\mathcal{M}',s') \models pre(e)$. Let $\mathcal{B} : (\mathcal{M},s) \leftrightarrows (\mathcal{M}',s')$.
Then the bisimulation $\mathcal{B}' : (\mathcal{M}\otimes M,(s,e)) \leftrightarrows (\mathcal{M}'\otimes M,(s',e))$ between the successor states can be defined as
$\mathcal{B}'((t,\varepsilon),(t',\varepsilon'))$ iff $\mathcal{B}(t,t')$ and $\varepsilon = \varepsilon'$ for all (t,ε) and (t',ε'). □

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Definition (Bisimulation of actions)

Let two pointed action models (M,ℓ) with $M = (E,\sim,pre)$ and (M',ℓ') with $M' = (E',\sim',pre')$ be given. A non-empty relation $\mathcal{B} \subseteq E \times E'$ is a bisimulation between (M,ℓ) and (M',ℓ') iff $\mathcal{B}(\ell,\ell')$, and for all $e \in E$ and $e' \in E'$ with $\mathcal{B}(e,e')$, the following holds:

- (forth) for all agents $a \in A$ and $\varepsilon \in E$, if $e \sim_a \varepsilon$, then there is an $\varepsilon' \in E'$ such that $e' \sim'_a \varepsilon'$ and $\mathcal{B}(\varepsilon,\varepsilon')$,
- (back) for all agents $a \in A$ and $\varepsilon' \in E'$, if $e' \sim'_a \varepsilon'$, then there is an $\varepsilon \in E$ such that $e \sim_a \varepsilon$ and $\mathcal{B}(\varepsilon,\varepsilon')$, and
- (pre) $pre(e)$ and $pre'(e')$ are logically equivalent.

Bisimilarity and Action Emulation

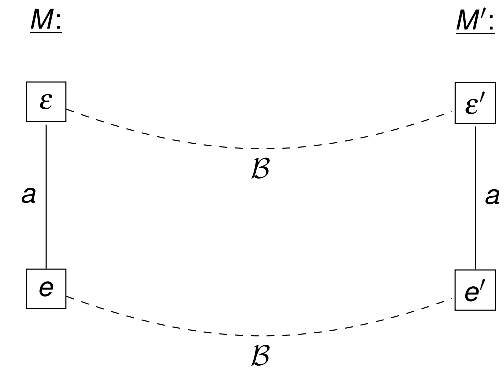
Definition (Bisimulation of actions, ctd.)

$\mathcal{B}$ is a total bisimulation if for each $e \in E$, there is an $e' \in E'$ such that $\mathcal{B}$ is a bisimulation between (M, e) and (M', e') and vice versa.

We write $(M, e) \Leftrightarrow (M', e')$ iff there is a bisimulation between M and M' linking e and e', and we then say that (M, e) and (M', e') are bisimilar.

Bisimilarity and Action Emulation

Forth condition, visualized



Bisimilarity and Action Emulation

Now, can we prove that bisimilar action models are always interchangeable? Yes!

Proposition

Given two action models $(M, e) \Leftrightarrow (M', e')$ and an epistemic state $(\mathcal{M}, s)$ such that (M, e) is applicable in $(\mathcal{M}, s)$. Then $(\mathcal{M} \otimes M, (s, e)) \Leftrightarrow (\mathcal{M} \otimes M', (s, e'))$.

Proof.

Let $\mathcal{B} : (M, e) \Leftrightarrow (M', e')$. Then $\models pre'(e') \leftrightarrow pre(e)$, because $\mathcal{B}(e, e')$. Since (M, e) is applicable in $(\mathcal{M}, s)$, we have $\mathcal{M}, s \models pre(e)$. Hence, also $\mathcal{M}, s \models pre'(e')$, i. e., (M', e') is also applicable in $(\mathcal{M}, s)$. [...]

Bisimilarity and Action Emulation

Proof (ctd.)

The bisimulation $\mathcal{B}' : (\mathcal{M} \otimes M, (s, e)) \Leftrightarrow (\mathcal{M} \otimes M', (s, e'))$ is defined as $\mathcal{B}'((t, \varepsilon), (t', \varepsilon'))$ iff $t = t'$ and $\mathcal{B}(\varepsilon, \varepsilon')$.
The forth and back conditions follow from those of $\mathcal{B}$.
Valuations: $t \in V_p$ iff $t' \in V_p$, and ε and ε' do not affect the valuations. □

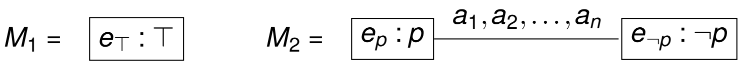
Proposition

If $(\mathcal{M}, s) \Leftrightarrow (\mathcal{M}', s')$ and $(M, e) \Leftrightarrow (M', e')$, then also $(\mathcal{M} \otimes M, (s, e)) \Leftrightarrow (\mathcal{M}' \otimes M', (s', e'))$. □

Bisimilarity and Action Emulation

Example

Recall our earlier example. M_1 and M_2 are not bisimilar, but always behave in the same way $\rightsquigarrow$ similar enough.



Question: How to formalize "similar enough"?

Answer: Action emulation!

Bisimilarity and Action Emulation

Definition (Action emulation)

Let two pointed action models (M, ℓ) with $M = (E, \sim, pre)$ and (M', ℓ') with $M' = (E', \sim', pre')$ be given. An emulation between (M, ℓ) and (M', ℓ') is a relation $\mathcal{E} \subseteq E \times E'$ such that $\mathcal{E}(\ell, \ell')$, and for all $a \in A$, all $e, \varepsilon \in E$ and all $e', \varepsilon' \in E'$, the following holds:

- (forth) if $\mathcal{E}(e, e')$ and $e \sim_a \varepsilon$, then there are $\varepsilon'_1, \ldots, \varepsilon'_n \in E'$ such that for all $i = 1, \ldots, n$, $\mathcal{E}(\varepsilon, \varepsilon'_i)$ and $e' \sim'_a \varepsilon'_i$, and $pre(\varepsilon) \models pre'(\varepsilon'_1) \vee \cdots \vee pre'(\varepsilon'_n)$.
- (back) if $\mathcal{E}(e, e')$ and $e' \sim'_a \varepsilon'$, then there are $\varepsilon_1, \ldots, \varepsilon_n \in E$ such that for all $i = 1, \ldots, n$, $\mathcal{E}(\varepsilon_i, \varepsilon')$ and $e \sim_a \varepsilon_i$, and $pre'(\varepsilon') \models pre(\varepsilon_1) \vee \cdots \vee pre(\varepsilon_n)$.
- (pre) if $\mathcal{E}(e, e')$, then $pre(e) \wedge pre'(e')$ is consistent (unless $pre(e)$ or $pre'(e')$ is already inconsistent).

Bisimilarity and Action Emulation

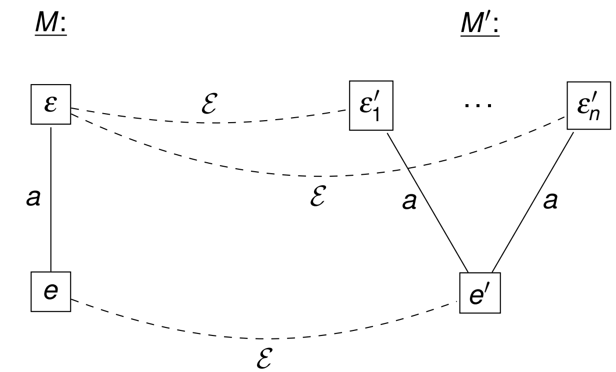
Definition (Action emulation, ctd.)

$\mathcal{E}$ is a total emulation if for each $e \in E$, there is an $e' \in E'$ with $\mathcal{E}(e, e')$ and vice versa.

We write $\mathcal{E} : (M, e) \rightleftarrows (M', e')$ iff there is an emulation $\mathcal{E}$ between M and M' linking e and e', and we then say that (M, e) and (M', e') are emulous.

Bisimilarity and Action Emulation

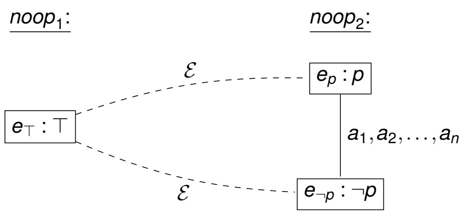
Forth condition, visualized



$pre(\varepsilon) \models pre'(\varepsilon'_1) \vee \cdots \vee pre'(\varepsilon'_n)$

Bisimilarity and Action Emulation

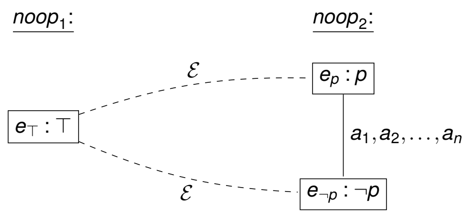
Example (Action emulation)



- Emulation: $\mathcal{E} = \{(e_\top, e_p), (e_\top, e_{\neg p})\}$.
- Forth: For $\mathcal{E}(e_\top, e_p)$: only $e_\top \sim_a e_\top$ for all $a \in A$. Need to find events $\varepsilon'_1, \dots, \varepsilon'_n$ such that $\mathcal{E}(e_\top, \varepsilon'_i)$ and $e_p \sim'_a \varepsilon'_i$ for all $i = 1, \dots, n$, and $pre(e_\top) \models pre'(\varepsilon'_1) \vee \dots \vee pre'(\varepsilon'_n)$. Choose $\{\varepsilon'_1, \dots, \varepsilon'_n\} = \{e_p, e_{\neg p}\}$. Then $pre(e_\top) = \top \models p \vee \neg p = pre'(e_p) \vee pre'(e_{\neg p})$. $\mathcal{E}(e_\top, e_{\neg p})$ similar.

Bisimilarity and Action Emulation

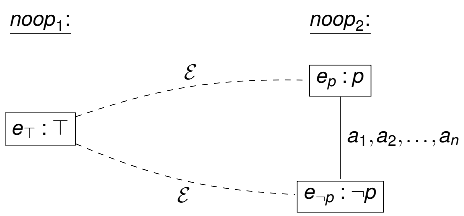
Example (Action emulation, ctd.)



- Back: Exemplarily for $\mathcal{E}(e_\top, e_p)$ ($\mathcal{E}(e_\top, e_{\neg p})$ similar): we have $e_p \sim'_a e_p$ and $e_p \sim'_a e_{\neg p}$ for all agents a. Exemplarily for $e_p \sim'_a e_p$ (again, $e_p \sim'_a e_{\neg p}$ similar). Need to find events $\varepsilon_1, \dots, \varepsilon_n$ such that $\mathcal{E}(\varepsilon_i, e_p)$ and $e_\top \sim_a \varepsilon_i$ for all $i = 1, \dots, n$, and $pre'(e_p) \models pre(\varepsilon_1) \vee \dots \vee pre(\varepsilon_n)$. Choose $\{\varepsilon_1, \dots, \varepsilon_n\} = \{e_\top\}$. Then $pre'(e_p) = p \models \top = pre(e_\top)$.

Bisimilarity and Action Emulation

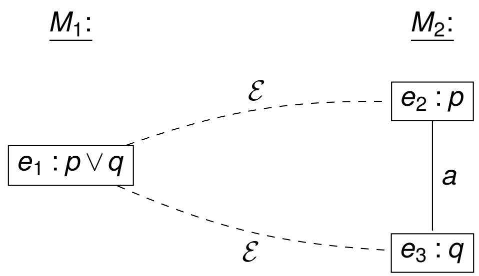
Example (Action emulation, ctd.)



- Pre:
  - For $(e_\top, e_p)$: $pre(e_\top) \wedge pre'(e_p) = \top \wedge p \equiv p$ is consistent.
  - For $(e_\top, e_{\neg p})$: $pre(e_\top) \wedge pre'(e_{\neg p}) = \top \wedge \neg p \equiv \neg p$ is consistent.

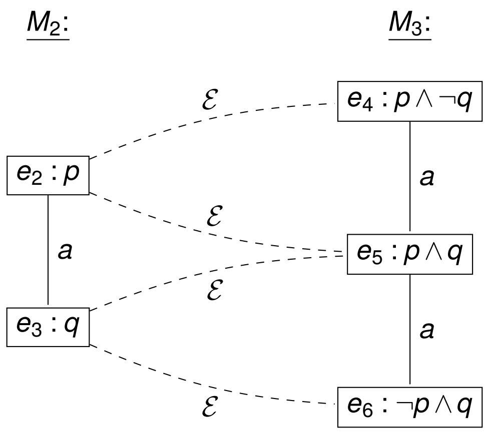
Bisimilarity and Action Emulation

Example (Action emulation, Ex. 2)



Bisimilarity and Action Emulation

Example (Action emulation, Ex. 3)



Bisimilarity and Action Emulation

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Proposition (Bisimulations are emulations)

A bisimulation $\mathcal{B} : (M, e) \leftrightarrows (M', e')$ is also an emulation.

Proof.

Easy. Homework. □

Bisimilarity and Action Emulation

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Proposition (Emulation guarantees bisimilarity)

Given an epistemic model $\mathcal{M}$ and action models $M \rightleftarrows M'$. Then $\mathcal{M} \otimes M \leftrightarrows \mathcal{M} \otimes M'$.

Proof.

Let $M = (E, \sim, pre)$ and $M' = (E', \sim', pre')$ with $\mathcal{E} : M \rightleftarrows M'$. Define $\mathcal{B} : \mathcal{M} \otimes M \leftrightarrows \mathcal{M} \otimes M'$ as $\mathcal{B}((s,e),(s',e'))$ iff $s = s'$ and $\mathcal{E}(e,e')$. Show that $\mathcal{B}$ is a total bisimulation between $\mathcal{M} \otimes M$ and $\mathcal{M} \otimes M'$.

- Forth: Let $(s,e) \sim_a (t,\varepsilon)$ and $\mathcal{B}((s,e),(s,e'))$. Then $s \sim_a t$, $e \sim_a \varepsilon$, and $\mathcal{E}(e,e')$.
  Therefore, there are events $\varepsilon'_1, \dots, \varepsilon'_n$ such that $\mathcal{E}(\varepsilon, \varepsilon'_1), \dots, \mathcal{E}(\varepsilon, \varepsilon'_n)$ and $e' \sim'_a \varepsilon'_1, \dots, e' \sim'_a \varepsilon'_n$, and $pre(\varepsilon) \models pre'(\varepsilon'_1) \vee \dots \vee pre'(\varepsilon'_n)$. [...]

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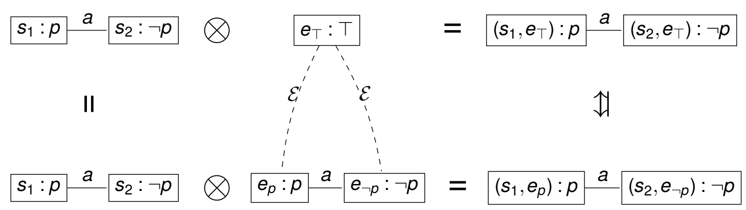
Proof (ctd.)

- Forth: [...] We know that $(t,\varepsilon) \in \mathcal{D}(\mathcal{M} \otimes M)$. So, $\mathcal{M}, t \models pre(\varepsilon)$, and hence $\mathcal{M}, t \models pre'(\varepsilon'_1) \vee \dots \vee pre'(\varepsilon'_n)$. So, there is an $1 \leq i \leq n$ such that $\mathcal{M}, t \models pre'(\varepsilon'_i)$. Therefore, $(t, \varepsilon'_i) \in \mathcal{D}(\mathcal{M} \otimes M')$. Furthermore, $\mathcal{B}((t,\varepsilon),(t,\varepsilon'_i))$ by definition of $\mathcal{B}$, and $(s,e') \sim_a (t,\varepsilon'_i)$, since $s \sim_a t$ and $e' \sim'_a \varepsilon'_i$.
- Back: Similar.
- Valuations: $\mathcal{B}((s,e),(s',e'))$ implies $s = s'$. Action applications do not affect the valuations. □

Remark: For action models with propositional preconditions, action emulation fully characterizes the effect of action application. I. e., if $\mathcal{M} \otimes M \leftrightarrows \mathcal{M} \otimes M'$, then $M \rightleftarrows M'$.

Bisimilarity and Action Emulation

Example (Emulation guarantees bisimilarity)



Validities and Axiomatisation

Validities and Axiomatisation

Recall: In public announcement logic, $\langle\psi\rangle\varphi \rightarrow [\psi]\varphi$ is valid.

Question: Is $\langle\alpha\rangle\varphi \rightarrow [\alpha]\varphi$ also valid in action model logic?

Answer: No!

Reason: Nondeterminism. Potentially, after some outcome of α, φ is true, but not after every outcome of α.

Counterexample: $\varphi = K_a p$ and
$\alpha = \text{Mayread} = (\text{Mayread}, e_1) \cup (\text{Mayread}, e_2) \cup (\text{Mayread}, e_3)$.
(After the outcome of Mayread in which Alice reads p, she knows p, but after the outcome where she does not read the letter, she does not know p).

Validities and Axiomatisation

But: $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$ is valid.

$\rightsquigarrow$ get rid of nondeterminism.

$\rightsquigarrow$ assume no nondeterminism for the rest of this section.

$\rightsquigarrow$ justification for formulating all principles of action model logic in terms of action models only (no nondeterministic choice).

Validities and Axiomatisation

Proposition (atomic permanence)

$[M,e]p \leftrightarrow (pre(e) \rightarrow p)$ is valid. □

Proposition (action and negation)

$[M,e]\neg\varphi \leftrightarrow (pre(e) \rightarrow \neg[M,e]\varphi)$ is valid. □

Proposition (action and conjunction)

$[M,e](\varphi \wedge \psi) \leftrightarrow ([M,e]\varphi \wedge [M,e]\psi)$ is valid. □

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Question: What about public announcement logic principle/validity $[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$?

Answer: It does not directly generalise to action model logic.

That is, the formula $[M,e]K_a\psi \leftrightarrow (pre(e) \rightarrow K_a[M,e]\psi)$ is not valid (not even for deterministic $\alpha = (M,e)$!)

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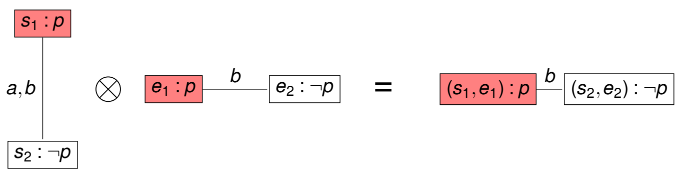
Example (Model $(\text{Before}, s_1) \otimes (\text{Read}, e_1)$)



On the other hand:

- $\text{Before}, s_1 \not\models [\text{Read}, e_1]K_b p$ since
  - $\text{Before}, s_1 \models pre(e_1)$, but
  - $\text{Before} \otimes \text{Read}, (s_1, e_1) \not\models K_b p$.

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Example (Model $(\text{Before}, s_1) \otimes (\text{Read}, e_1)$, ctd.)

$\text{Before}, s_1 \not\models [\text{Read}_a, e_1]K_b p \leftrightarrow (pre(e_1) \rightarrow K_b[\text{Read}_a, e_1]p)$.

Hence, $[M,e]K_a\varphi \leftrightarrow (pre(e) \rightarrow K_a[M,e]\varphi)$ is not valid!

Intuition: Agent b may mistake action (Read_a, e_1) for action (Read_a, e_2) when observing it. Hence, when observing (Read_a, e_1), he does not learn that p is true, but also considers it possible that agent a just learned $\neg p$.

Remark: Agent b does observe that (Read_a, e_1) or (Read_a, e_2) happens; he just cannot distinguish between them.

Hypothetically, if for both actions (Read_a, e_1) and (Read_a, e_2), agent b knew that they produce p, then after (Read_a, e_1), agent b would also know that p is true.

Validities and Axiomatisation

This provides intuition for the following proposition:

Proposition (action and knowledge)

$[M,e]K_a\varphi \leftrightarrow (pre(e) \rightarrow \bigwedge_{e\sim_a\varepsilon} K_a[M,\varepsilon]\varphi)$ is valid.

Proof.

We prove the dual: $\langle M,e\rangle\hat{K}_a\varphi \leftrightarrow (pre(e) \wedge \bigvee_{e\sim_a\varepsilon} \hat{K}_a\langle M,\varepsilon\rangle\varphi)$ is valid. Let $\mathcal{M} = (S,\sim,V)$ and $M = (E,\sim,pre)$.

- ($\Rightarrow$) Assume that $\mathcal{M},s \models \langle M,e\rangle\hat{K}_a\varphi$. Then $\mathcal{M},s \models pre(e)$ and $\mathcal{M}\otimes M,(s,e) \models \hat{K}_a\varphi$. Then there is a $(t,\varepsilon) \in S\times E$ such that $(s,e) \sim_a (t,\varepsilon)$ and $\mathcal{M}\otimes M,(t,\varepsilon) \models \varphi$. Thus, $s \sim_a t$ and $e \sim_a \varepsilon$. Moreover, $\mathcal{M},t \models \langle M,\varepsilon\rangle\varphi$. With $s \sim_a t$, we get $\mathcal{M},s \models \hat{K}_a\langle M,\varepsilon\rangle\varphi$.
  So, with $e \sim_a \varepsilon$, we get $\mathcal{M},s \models \bigvee_{e\sim_a\varepsilon} \hat{K}_a\langle M,\varepsilon\rangle\varphi$.
- ($\Leftarrow$) [. . .]

Validities and Axiomatisation

Proof (ctd.)

We prove the dual: $\langle M,e\rangle\hat{K}_a\varphi \leftrightarrow (pre(e) \wedge \bigvee_{e\sim_a\varepsilon} \hat{K}_a\langle M,\varepsilon\rangle\varphi)$ is valid. Let $\mathcal{M} = (S,\sim,V)$ and $M = (E,\sim,pre)$.

- ($\Rightarrow$) [. . .]
- ($\Leftarrow$) Assume that $\mathcal{M},s \models pre(e)$ and there is an event $\varepsilon \in E$ with $e \sim_a \varepsilon$ and $\mathcal{M},s \models \hat{K}_a\langle M,\varepsilon\rangle\varphi$. Then, $(s,e) \in \mathcal{D}(\mathcal{M}\otimes M)$ and there is a state $t \in S$ with $s \sim_a t$ and $\mathcal{M},t \models \langle M,\varepsilon\rangle\varphi$. Thus $\mathcal{M},t \models pre(\varepsilon)$, and $(t,\varepsilon) \in \mathcal{D}(\mathcal{M}\otimes M)$, and $(\mathcal{M}\otimes M,(t,\varepsilon)) \models \varphi$. With $s \sim_a t$ and $e \sim_a \varepsilon$, we get $(s,e) \sim_a (t,\varepsilon)$. Hence, $\mathcal{M}\otimes M,(s,e) \models \hat{K}_a\varphi$.
  So, $\mathcal{M},s \models \langle M,e\rangle\hat{K}_a\varphi$. □

Validities and Axiomatisation

Proposition (Actions and common knowledge)

Given an action model (M,e) and formulas χ_ε for all $\varepsilon \sim_B e$. If for all $a \in B$ and for all $\ell \sim_a \varepsilon$, $\models \chi_\varepsilon \rightarrow [M,\varepsilon]\varphi$ and $\models (\chi_\varepsilon \wedge pre(\varepsilon)) \rightarrow K_a\chi_\ell$, then $\models \chi_e \rightarrow [M,e]C_B\varphi$.

Proof.

Let $M = (E,\sim,pre)$. We need to show $\models \chi_e \rightarrow [M,e]C_B\varphi$. Assume an arbitrary $(\mathcal{M},s)$ such that $\mathcal{M},s \models \chi_e$, and assume that $\mathcal{M},s \models pre(e)$. Then we need to show that $(\mathcal{M}\otimes M,(s,e)) \models C_B\varphi$. Assume an arbitrary state $(u,\ell) \in \mathcal{D}(\mathcal{M}\otimes M)$ that is B-accessible from (s,e). We show that $(\mathcal{M}\otimes M,(u,\ell)) \models \varphi$ by induction on the path length from (s,e) to (u,ℓ). We prove the stronger statement $(\mathcal{M}\otimes M,(u,\ell)) \models \varphi$ and $\mathcal{M},u \models \chi_\ell$. [. . .]

Validities and Axiomatisation

Proof (ctd.)

- Base case ($n = 0$): Follows from $\models \chi_\varepsilon \rightarrow [M,\varepsilon]\varphi$ for $\varepsilon = e$, applied to $\mathcal{M},s$, and from assumptions $\mathcal{M},s \models \chi_e$ and $\mathcal{M},s \models pre(e)$.
- Inductive case (from n to $n+1$): There is a state (t,ε) such that $(s,e) \sim_B (t,\varepsilon) \sim_a (u,\ell)$, where the path linking (s,e) to (t,ε) has length n. With the induction hypothesis, we get $(\mathcal{M}\otimes M,(t,\varepsilon)) \models \varphi$ and $\mathcal{M},t \models \chi_\varepsilon$. With $\mathcal{M},t \models \chi_\varepsilon$ and $\mathcal{M},t \models pre(\varepsilon)$ and assumption $\models (\chi_\varepsilon \wedge pre(\varepsilon)) \rightarrow K_a\chi_\ell$, we get $\mathcal{M},t \models K_a\chi_\ell$. With $t \sim_a u$, we get $\mathcal{M},u \models \chi_\ell$. With assumed validity $\models \chi_\varepsilon \rightarrow [M,\varepsilon]\varphi$, we get $\mathcal{M},u \models [M,\ell]\varphi$. With $(u,\ell) \in \mathcal{D}(\mathcal{M}\otimes M)$, we get $\mathcal{M},u \models pre(\ell)$. Hence, $(\mathcal{M}\otimes M,(u,\ell)) \models \varphi$. □

Validities and Axiomatisation

AMC

Axioms and inference rules for action model logic **AMC**:

- all axioms and rules of **S5C** with common knowledge
- $[M,e]p \leftrightarrow (pre(e) \rightarrow p)$ (Atomic permanence)
- $[M,e]\neg\varphi \leftrightarrow (pre(e) \rightarrow \neg[M,e]\varphi)$ (Action + negation)
- $[M,e](\varphi \wedge \psi) \leftrightarrow ([M,e]\varphi \wedge [M,e]\psi)$ (Action + conj.)
- $[M,e]K_a\varphi \leftrightarrow (pre(e) \rightarrow \bigwedge_{e \sim_a \varepsilon} K_a[M,\varepsilon]\varphi)$ (Action + knowl.)
- $[M,e][M',e']\varphi \leftrightarrow [(M,e);(M',e')]\varphi$ (Composition)
- $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$ (Nondeterministic choice)
- From φ, infer $[M,e]\varphi$ (Neccessitation of $[M,e]$)
- Given action model (M,e) and χ_ε for all $\varepsilon \sim_B e$. If for all $a \in B$ and $\ell \sim_a \varepsilon$, $\chi_\varepsilon \rightarrow [M,\varepsilon]\varphi$ and $(\chi_\varepsilon \wedge pre(\varepsilon)) \rightarrow K_a\chi_\ell$, then infer $\chi_e \rightarrow [M,e]C_B\varphi$ (Action + common knowledge)

Validities and Axiomatisation

Theorem

The axiomatisation **AMC** *is sound and complete for the set of all valid formulas in* $\mathcal{L}_{KC\otimes}$. □

Validities and Axiomatisation

Example

We show that $\vdash [\mathrm{Read}_a, e_1]K_a p$:

1. $p \rightarrow p$ (prop. taut.)
2. $[\mathrm{Read}_a, e_1]p \leftrightarrow (p \rightarrow p)$ (atomic permanence, $pre(e_1) = p$)
3. $[\mathrm{Read}_a, e_1]p$ (1, 2, prop. reasoning)
4. $K_a[\mathrm{Read}_a, e_1]p$ (3, necc. of K_a)
5. $p \rightarrow K_a[\mathrm{Read}_a, e_1]p$ (4, prop. reasoning, weakening)
6. $[\mathrm{Read}_a, e_1]K_a p \leftrightarrow (p \rightarrow \bigwedge_{\varepsilon \sim_a e_1} K_a[\mathrm{Read}_a, \varepsilon]p)$ (action + knowledge, $[e_1]_{\sim_a} = \{e_1\}$)
7. $[\mathrm{Read}_a, e_1]K_a p$ (5, 6, prop. reasoning)

Summary

Summary

- Action models allow more epistemic change than just public announcements.
- Action models similar to Kripke structures. State update by product update operator.
- Emulous action models are interchangeable.
- Axiomatization similar to public announcement logic. Actions and (common) knowledge slightly trickier.