Dynamic Epistemic Logic

2. The Multi-Agent S5 System

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Language

Basic Epistemic Language

When we want to define the basic epistemic language, we need sets of agent symbols and sets of atomic propositions to talk about. Specifically, we have:

- a finite set A of agent symbols (often: a, b, a', a'', ...)
- a countable set P of atomic propositions (often: p, q, p', p'', ...)

Definition (Basic epistemic language)

Let P be a countable set of atomic propositions and A be a finite set of agent symbols. Then the language $\mathcal{L}_K$ is defined by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi,$$

where $p \in P$ and $a \in A$.

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Basic Epistemic Language

We use some common abbreviations and conventions:

- $(\varphi \vee \psi) = \neg(\neg\varphi \wedge \neg\psi)$
- $(\varphi \rightarrow \varphi) = (\neg\varphi \vee \psi)$
- $(\varphi \leftrightarrow \psi) = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- $\top = p \vee \neg p$ for some $p \in P$
- $\bot = \neg\top$

If there is no risk of confusion, outer parentheses can be omitted.

Basic Epistemic Language

Only interesting addition compared to propositional logic: the knowledge modalities K_a.

- $K_a\varphi$ is read as "agent a knows that φ (is true)".
- Its dual, $\neg K_a \neg\varphi$ is read as "agent a considers φ as possible". Abbreviation: $\hat{K}_a\varphi$.
- For a group of agents $B \subseteq A$, we write $E_B\varphi$ to express that everybody in B knows φ. I. e., $E_B\varphi \equiv \bigwedge_{b\in B} K_b\varphi$.
- Its dual is $\hat{E}_B\varphi = \neg E_B \neg\varphi \equiv \bigvee_{b\in B} \hat{K}_b\varphi$, which can be read as "some agent b in B considers φ as possible".
- Sometimes, when writing *iterated operators*, the following convention comes in handy: if X is a modal operator, then X^n is the n-fold application of X. E. g., $K_a^3\varphi$ means $K_aK_aK_a\varphi$.

Example (Simplified Hanabi)

In simplified Hanabi, we have four cards ($r1$, $r2$, $g1$, $g2$), two players (a, b), and just one card per player. We write p_c for the fact that player p holds card c. Thus, for instance, a_{r1} is read as "player a has card $r1$". Consider the situation where player a has card $r1$ and player b has card $r2$. In this situation, all of the following formulas are true:

- a_{r1} and b_{r2},
- $K_a b_{r2}$ and $K_b a_{r1}$,
- $K_a \neg a_{r2}$ and $K_b \neg b_{r1}$ (Notice that, to arrive at this conclusion, we need to make use of our background theory that contains assertions such as $\neg(a_{r1} \wedge b_{r1})$),
- $K_a(K_b a_{r1} \vee K_b a_{g1} \vee K_b a_{g2})$.

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Semantics

Kripke Models

The semantics of the basic epistemic language is based on a special form of Kripke semantics, where we have

- states (or worlds),
- accessibility relations (or indistinguishability relations) between the worlds, and
- propositional valuations associated with the worlds.

Example (Kripke models)

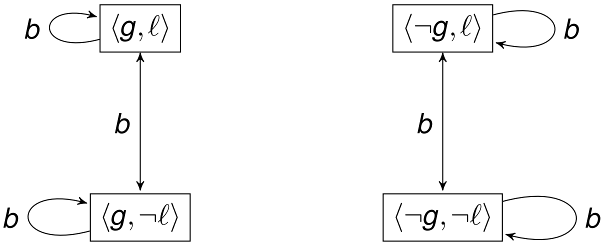
Consider two cities, namely Groningen and Liverpool. Assume that:

- Person *b* lives in Groningen.
- Person *w* lives in Liverpool.
- "The weather in Groningen is sunny" is the atomic proposition g.
- "The weather in Liverpool is sunny" is the atomic proposition ℓ.

States are just possible weather conditions: $\langle g, \ell \rangle$, $\langle \neg g, \ell \rangle$, $\langle g, \neg \ell \rangle$, $\langle \neg g, \neg \ell \rangle$. We want to model what agent *b* knows. Assume that *b* is in state $\langle g, \ell \rangle$. He also considers the state $\langle g, \neg \ell \rangle$ possible.

Example (Kripke models (ctd.))

This situation can be graphically captured by the following model $\mathcal{M}_1$:



Definition (Kripke model)

Given a countable set of atomic propositions P and a finite set of agent names A, a Kripke model is a structure $\mathcal{M} = (S, R_A, V_P)$ where:

- S is a set of states (also called the domain of $\mathcal{M}$, in symbols $\mathcal{D}(\mathcal{M})$),
- R_A is a function yielding, for every $a \in A$, an accessibility relation $R_A(a) = R_a \subseteq S \times S$.
- $V_P : P \to 2^S$ is a *valuation function* that for all $p \in P$ yields the set of worlds $V_P(p) \subseteq S$ where p is true.

Kripke Models

- If A and P are not important or clear from the context, we will often drop them and write $\mathcal{M} = (S, R, V)$.
- If all accessibility relations R_a are equivalence relations (reflexive, symmetric and transitive), then we also use the symbols $\sim$ for R and $\sim_a$ for R_a.
- In that case, $\mathcal{M} = (S, \sim, V)$ is also called an epistemic model.

Kripke Semantics

Formulas are then interpreted over states in models (aka. states, pointed models, epistemic states).

Example

- Assume we have the formula $K_b\ell$.
- This formula is *not* true in state $\langle\neg g, \ell\rangle$, symbolically $\langle\neg g, \ell\rangle \not\models K_b\ell$.
- Reason: In $\langle\neg g, \ell\rangle$, agent b also considers world $\langle\neg g, \neg\ell\rangle$ possible, and in that world, ℓ does not hold.

We can define truth of an epistemic formula in an epistemic state inductively as follows.

Definition

Given a Kripke model $\mathcal{M} = (S, R, V)$ and $s \in S$, the pair $(\mathcal{M}, s)$ is called a pointed model. If $\mathcal{M}$ is an epistemic model, then $(\mathcal{M}, s)$ is called an epistemic state.

Definition

A formula φ is true in an epistemic state $(\mathcal{M}, s)$, symbolically $\mathcal{M}, s \models \varphi$, under the following conditions:

$$
\begin{array}{lll}
\mathcal{M}, s \models p & \text{iff} & s \in V(p) \\
\mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \not\models \varphi \\
\mathcal{M}, s \models K_a\varphi & \text{iff} & \mathcal{M}, t \models \varphi \text{ for all } t \in S \text{ with } (s,t) \in R_a
\end{array}
$$

This implies, among others, that $\mathcal{M}, s \models \hat{K}_a\varphi$ iff $\mathcal{M}, t \models \varphi$ for some $t \in S$ with $(s,t) \in R_a$.

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Definition

If $\mathcal{M}, s \models \varphi$ for all $s \in \mathcal{D}(\mathcal{M})$, then we say that φ is true in $\mathcal{M}$, symbolically, $\mathcal{M} \models \varphi$.

Definition

If $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ in a certain class $\mathcal{X}$ of models, then we say that φ is valid in $\mathcal{X}$, symbolically, $\mathcal{X} \models \varphi$.

Example

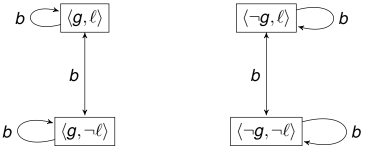
If φ is valid in the class $\mathcal{K}$ of all Kripke models, then we write $\mathcal{K} \models \varphi$.

Definition

If there exists a pointed model $(\mathcal{M}, s)$ such that φ is true in $(\mathcal{M}, s)$, then we say φ is satisfied in $(\mathcal{M}, s)$. If $\mathcal{M}$ belongs to a class of models $\mathcal{X}$, then φ is satisfiable in $\mathcal{X}$.

Example

Recall model $\mathcal{M}_1$:



- $\mathcal{M}_1, \langle g, \ell \rangle \models K_b g$
- $\mathcal{M}_1, \langle g, \ell \rangle \models \neg K_b \ell$
- $\mathcal{M}_1, \langle g, \ell \rangle \models \neg K_b \neg \ell$
- $\rightsquigarrow \mathcal{M}_1, \langle g, \ell \rangle \models K_b g \wedge \neg K_b \ell \wedge \neg K_b \neg \ell$.

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Example (Higher-order knowledge)

$\mathcal{M}_1, \langle g, \ell \rangle \models K_b(K_b g \wedge \neg K_b \ell)$.

To see this, we have to verify that:

- $\mathcal{M}_1, \langle g, \ell \rangle \models K_b g \wedge \neg K_b \ell$.
- $\mathcal{M}_1, \langle g, \neg\ell \rangle \models K_b g \wedge \neg K_b \ell$.

In both cases, agent b considers the same states as possible, namely $\langle g, \ell \rangle$ and $\langle g, \neg\ell \rangle$.

- $K_b g$ is true because in all accessible states, g is true.
- $\neg K_b \ell$ is true because there is an accessible state, namely $\langle g, \neg\ell \rangle$, where ℓ is not true.

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Example

$\mathcal{M}_1 \models (K_b g \vee K_b \neg g) \wedge (\neg K_b \ell \wedge \neg K_b \neg l)$.
Easy to see that both clauses are true and thus the whole formula is true.

Convention

From now on: Visualizations of epistemic models use undirected edges and leave out reflexive and transitive edges.

Kripke Semantics

Example

Model $\mathcal{M}_2$:

$$
\begin{tikzcd}
\boxed{\langle g, \ell \rangle} \arrow[r, dash, "w"] \arrow[d, dash, "b"'] & \boxed{\langle \neg g, \ell \rangle} \arrow[d, dash, "b"] \\
\boxed{\langle g, \neg\ell \rangle} \arrow[r, dash, "w"] & \boxed{\langle \neg g, \neg\ell \rangle}
\end{tikzcd}
$$

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Example

- $\mathcal{M}_2, \langle g, \ell \rangle \models (K_b g \vee K_b \neg g) \wedge (K_w \ell \vee K_w \neg \ell)$
  (agent b knows whether g, and w knows whether ℓ).
- $\mathcal{M}_2, \langle g, \ell \rangle \models \neg K_w g \wedge \neg K_w \neg g \wedge K_w (K_b g \vee K_b \neg g)$
  (although agent b is ignorant about g, he knows that agent w actually knows whether g holds).

Question: Can we also come up with a model that describes ignorance about what the other knows?

Answer: Yes, but to do that we need to introduce more worlds. Note that there can be distinct states with identical valuations!

Kripke Semantics

Example

Another agent *h* (from Otago, NZ) calls *w* on the phone. *w* tells *h* that ℓ is true. Then *h* tells *w* that he will call *b* afterwards, but he does not say whether he will tell *b* about ℓ. So, *w* does not know whether *b* knows that ℓ is true.

Remark: The construction of the corresponding epistemic model basically means starting with the original model and updating it with a particular action, namely *h* calling *b*.

Kripke Semantics

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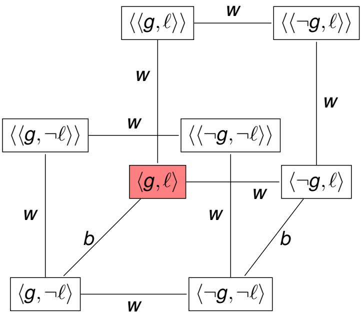
Example

Model $\mathcal{M}_2$:



$$\mathcal{M}_3, \langle g, \ell\rangle \models \ell \wedge \neg K_b \ell \wedge K_b(\neg K_w K_b \ell \wedge \neg K_w \neg K_b \ell)$$

Proposition

Let φ and ψ be formulas of $\mathcal{L}_K$ and let K_a be an epistemic operator for some $a \in A$. Let $\mathcal{K}$ be the set of all Kripke models and $\mathcal{S}5$ be the set of all epistemic models. Then the following hold:

- *(LO1)* $\mathcal{K} \models K_a\varphi \wedge K_a(\varphi \rightarrow \psi) \rightarrow K_a\psi$
- *(LO2)* $\mathcal{K} \models \varphi$ *implies* $\mathcal{K} \models K_a\varphi$
- *(LO3)* $\mathcal{K} \models \varphi \rightarrow \psi$ *implies* $\mathcal{K} \models K_a\varphi \rightarrow K_a\psi$
- *(LO4)* $\mathcal{K} \models \varphi \leftrightarrow \psi$ *implies* $\mathcal{K} \models K_a\varphi \leftrightarrow K_a\psi$
- *(LO5)* $\mathcal{K} \models (K_a\varphi \wedge K_a\psi) \rightarrow K_a(\varphi \wedge \psi)$
- *(LO6)* $\mathcal{K} \models K_a\varphi \rightarrow K_a(\varphi \vee \psi)$
- *(LO7)* $\mathcal{S}5 \models \neg(K_a\varphi \wedge K_a\neg\varphi)$

Definition (Relation properties)

A relation R is called

- reflexive if for all s, we have $(s,s) \in R$,
- symmetric if for all s,t, $(s,t) \in R$ implies $(t,s) \in R$,
- transitive if for all s,t,u, $(s,t) \in R$ and $(t,u) \in R$ implies $(s,u) \in R$,
- serial if for all s there is t such that $(s,t) \in R$,
- Euclidean if for all s,t,u, $(s,t) \in R$ and $(s,u) \in R$ implies $(t,u) \in R$, and
- an equivalence relation if it is reflexive, transitive, and symmetric (or: reflexive, transitive, and Euclidean).

Definition

Kripke models are classified according to the properties of the accessibility relation R_a as follows:

Relation property	**Name**
No restriction	$\mathcal{K}$
Serial	$\mathcal{KD}$
Reflexive	$\mathcal{T}$
Transitive	$\mathcal{K}4$
Reflexive and transitive	$\mathcal{S}4$
Transitive and Euclidean	$\mathcal{K}45$
Serial, transitive and Euclidean	$\mathcal{KD}45$
Serial, transitive, Euclidean and reflexive	$\mathcal{S}5$

Definition (Bisimulation)

Let two models $\mathcal{M} = (S, R, V)$ and $\mathcal{M}' = (S', R', V')$ be given. A non-empty relation $\mathcal{B} \subseteq S \times S'$ is a bisimulation iff for all $s \in S$ and $s' \in S'$ with $(s, s') \in \mathcal{B}$:

- (atoms) $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$,
- (forth) for all $a \in A$ and all $t \in S$, if $(s, t) \in R_a$, then there is a $t' \in S'$ such that $(s', t') \in R'_a$ and $(t, t') \in \mathcal{B}$, and
- (back) for all $a \in A$ and all $t' \in S'$, if $(s', t') \in R'_a$, then there is a $t \in S$ such that $(s, t) \in R_a$ and $(t, t') \in \mathcal{B}$.

We write $(\mathcal{M}, s) \leftrightarrows (\mathcal{M}', s')$ iff there is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ linking s and s', and we then say that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are bisimilar.

Bisimulations

The epistemic language $\mathcal{L}_K$ cannot distinguish between bisimilar models.

We write $(\mathcal{M},s) \equiv_{\mathcal{L}_K} (\mathcal{M}',s')$ if and only if $(\mathcal{M},s) \models \varphi$ iff $(\mathcal{M}',s') \models \varphi$ for all formulas $\varphi \in \mathcal{L}_K$.

Theorem (Bisimulation)

For all pointed models $(\mathcal{M},s)$ and $(\mathcal{M}',s')$, if $(\mathcal{M},s) \leftrightarroweq (\mathcal{M}',s')$, then $(\mathcal{M},s) \equiv_{\mathcal{L}_K} (\mathcal{M}',s')$.

Bisimulations

Proof.

By structural induction on φ. Suppose that $(\mathcal{M}, s) \leftrightarroweq (\mathcal{M}', s')$.

- Base case: For atomic formulas $\varphi = p \in P$, by atoms, it must be the case that $\mathcal{M}, s \models p$ iff $\mathcal{M}', s' \models p$ for all $p \in P$.
- Inductive cases: Given formula φ, assume that the claim is already proven for all strict subformulas φ' of φ.
  - Negation: Suppose that $\mathcal{M}, s \models \neg\varphi'$. By definition, this holds iff $\mathcal{M}, s \not\models \varphi'$. By induction hypothesis, this is equivalent to $\mathcal{M}', s' \not\models \varphi'$, which in turn is equivalent to $\mathcal{M}', s' \models \neg\varphi'$.

Proof (ctd.)

- Inductive cases: …
  - Conjunction: Suppose that $\mathcal{M}, s \models \varphi_1 \wedge \varphi_2$. By definition, this holds iff $\mathcal{M}, s \models \varphi_1$ and $\mathcal{M}, s \models \varphi_2$. By two applications of the induction hypothesis, this is equivalent to $\mathcal{M}', s' \models \varphi_1$ and $\mathcal{M}', s' \models \varphi_2$, which in turn is equivalent to $\mathcal{M}', s' \models \varphi_1 \wedge \varphi_2$.

Bisimulations

Proof (ctd.)

- Inductive cases: ...
  - Individual epistemic operators: Suppose that $\mathcal{M}, s \models K_a \varphi'$. Take an arbitrary t' such that $(s', t') \in R'_a$. By back, there is a state $t \in S$ such that $(s, t) \in R_a$ and $(t, t') \in \mathcal{B}$. With $(t, t') \in \mathcal{B}$ and by induction hypothesis, we get $\mathcal{M}, t \models \varphi'$ iff $\mathcal{M}', t' \models \varphi'$. Since $\mathcal{M}, s \models K_a \varphi'$ and $(s, t) \in R_a$, also $\mathcal{M}, t \models \varphi'$ must hold. Therefore, $\mathcal{M}', t' \models \varphi'$. Since t' was chosen arbitrarily from the states indistinguishable from s', it must be the case that $\mathcal{M}', t' \models \varphi'$ for all t' such that $(s', t') \in R'_a$. Therefore, by the semantics of knowledge operators, $\mathcal{M}', s' \models K_a \varphi'$.
    The opposite direction is similar, but the forth condition is used.

$\square$

Bisimulations

Remarks:

- $(\mathcal{M}, s) \leftrightarroweq (\mathcal{M}', s')$ implies $(\mathcal{M}, s) \equiv_{\mathcal{L}_K} (\mathcal{M}', s')$, but the converse does not hold.
- The proof applies to all classes of models, not only epistemic models.

Axiomatization

Axiomatization

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Logic = set of formulas

Possible ways of characterizing a logic and reasoning in it:

- Semantic derivation of valid formulas via Kripke models
- Syntatic derivation of valid formulas via axioms and inference rules

Axiomatization

Axioms and inference rules of minimal modal logic **K**:

- *(Prop)* all instantiations of propositional tautologies
- *(K)* $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$
  (Distribution of K_a over $\to$)
- *(MP)* From φ and $\varphi \to \psi$, infer ψ
  (Modus ponens)
- *(Nec)* From φ, infer $K_a\varphi$
  (Necessitation of K_a)

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Definition (Derivation)

Let **X** be an arbitrary axiomatisation with axioms $Ax_1, \ldots, Ax_n$ and rules $Ru_1, \ldots, Ru_k$, where each rule Ru_j, $1 \leq j \leq k$, is of the form "From $\varphi_1, \ldots, \varphi_{j_{ar}}$, infer φ_j". We call j_{ar} the arity of the rule. Then a derivation of a formula φ within **X** is a finite sequence $\varphi_1, \ldots, \varphi_m$ of formulas such that:

1. $\varphi_m = \varphi$ and
2. every φ_i in the sequence is:
   1. either an instance of one of the axioms $Ax_1, \ldots, Ax_n$,
   2. or else the result of the application of one of the rules $Ru_1, \ldots, Ru_k$ to j_{ar} formulas in the sequence that appear before φ_i.

Axiomatization

If there is a derivation for φ in **X**, the we write $\vdash_{\mathbf{X}} \varphi$, or, if the system **X** is clear from the context, just $\vdash \varphi$.

We then say that φ is a theorem of **X**.

Axiomatization

Logic **K** describes only (arbitrary) Kripke models, including models where R_a does not necessarily reflect knowledge.

Consider, e. g., model $\mathcal{M}$ below:

$$\begin{tikzcd} \boxed{s_1 : p} \arrow[r, "a"] & \boxed{s_2 : \neg p} \arrow[loop above, "a"] \end{tikzcd}$$

- $(\mathcal{M}, s_1) \models p$, but
- $(\mathcal{M}, s_1) \models K_a \neg p$.

$\rightsquigarrow$ this violates that knowledge should imply truth.

Axiomatization

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We would like a logic where something like $\neg(p \wedge K_a \neg p)$ is a theorem.

Semantically, we solved this by requiring epistemic models to have reflexive accessibility relations (among other requirements).

Syntatically, we can add axiom $K_a\varphi \rightarrow \varphi$.

Axiomatization

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Axioms and inference rules of **S5**:

- All axioms and rules of **K**
- *(T)* $K_a\varphi \rightarrow \varphi$
  (Truth)
- *(4)* $K_a\varphi \rightarrow K_aK_a\varphi$
  (Positive introspection)
- *(5)* $\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$
  (Negative introspection)

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Example

Proof of $\vdash_{\mathbf{S5}} K_aK_bp \rightarrow K_ap$:

1. $K_bp \rightarrow p$
   (axiom T)
2. $K_a(K_bp \rightarrow p)$
   (Necessitation of K_a, 1)
3. $K_a(K_bp \rightarrow p) \rightarrow (K_aK_bp \rightarrow K_ap)$
   (axiom K with $\varphi = K_bp$ and $\psi = p$)
4. $K_aK_bp \rightarrow K_ap$
   (Modus ponens, 2+3)

Axiomatization

Theorem

Axiom system **K** *is sound and complete w.r.t. the class $\mathcal{K}$ of all Kripke models, i. e., for every formula φ in $\mathcal{L}_K$, we have $\vdash_{\mathbf{K}} \varphi$ iff $\mathcal{K} \models \varphi$.* □

Theorem

Axiom system **S5** *is sound and complete w.r.t. the class $\mathcal{S}5$ of all epistemic models, i. e., for every formula φ in $\mathcal{L}_K$, we have $\vdash_{\mathbf{S5}} \varphi$ iff $\mathcal{S}5 \models \varphi$.* □

Common knowledge

Common Knowledge

Recall "everybody knows": $E_B\varphi \equiv \bigwedge_{b\in B} K_b\varphi$.

- E_B satisifes axiom T, but not (positive or negative) introspection.
- I. e., $E_B\varphi \to E_B E_B\varphi$ is not valid.
- E. g., if agents a and b are both (separately) told that p is true, $E_{ab}p$ is true but not $E_{ab}E_{ab}p$.
- So, how to model that everybody knows that everybody knows that ... that p?
- $\leadsto$ the common knowledge operator:
  For $B \subseteq A$, $C_B\varphi \equiv \varphi \wedge E_B\varphi \wedge E_B^2\varphi \wedge E_B^3\varphi \wedge \ldots$,
  where $E_B^n\varphi = \underbrace{E_B E_B \ldots E_B}_{n \text{ times}}\varphi$.

Common Knowledge

Notational conventions:

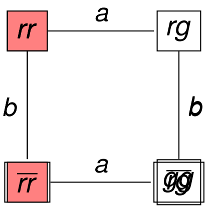
- Instead of $C_{\{a,b\}}$ or $E_{\{a,b\}}$, we often write C_{ab} and E_{ab}, respectively, etc.
- Instead of C_A or E_A, we usually write C and E, respectively, if A is the set of all agents.

Common Knowledge

Example (Common knowledge in card games)

Agents a and b are dealt one card each, both (independently) either red or green. They only see their own card. The actual card deal is rr. Now a tells b that she has a red card. Next, b leaves the room, giving a the chance to secretly look at b's card. She doesn't have to, but she does look.

Model $\mathcal{M}_1^{rg}$:

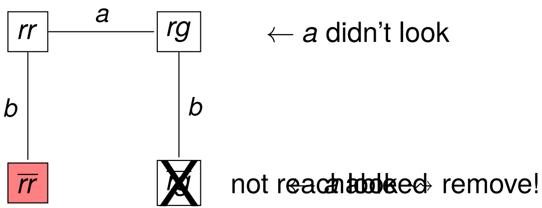


$\mathcal{M}_2^{rg}, rr \models C_{ab} red(a)$

Example (Common knowledge in card games, ctd.)

… She doesn't have to, but she does look. Now, *a* tells *b* that she looked at his card.

Model $\mathcal{M}_3^{rg}$:



Common Knowledge

By language $\mathcal{L}_{KC}$, we refer to the language defined like $\mathcal{L}_K$, but with the additional common knowledge modality C.

Definition (Epistemic language with common knowledge)

Let P be a countable set of atomic propositions and A be a finite set of agent symbols. Then the language $\mathcal{L}_{KC}$ is defined by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid C_B\varphi,$$

where $p \in P$, $a \in A$, and $B \subseteq A$.

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Semantics of common knowledge modality: as before, using (epistemic) Kripke models.

Definition (Accessibility relations for E_B and C_B)

Let $\mathcal{M} = (S, R, V)$ be a Kripke model with agents A and $B \subseteq A$.

- Then $R_{E_B} = \bigcup_{b \in B} R_b$.
- The transitive closure of a relation R is the smallest relation R^+ such that:
  - $R \subseteq R^+$, and
  - for all x, y, z, if $(x, y) \in R^+$ and $(y, z) \in R^+$ then also $(x, z) \in R^+$.

  If, additionally, $(x, x) \in R^+$ for all x, then R^+ is the reflexive-transitive closure of R, symbolically R^*.
- Then, define $R_{C_B} = R^*_{E_B}$. (Sometimes also $\sim_{C_B}$.)

Common Knowledge

Definition

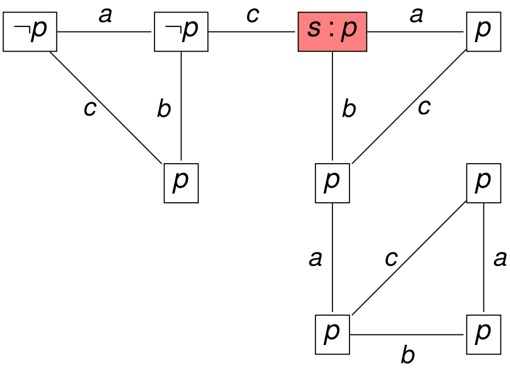
The truth of an $\mathcal{L}_{KC}$ formula φ in an epistemic state $(\mathcal{M}, s)$, symbolically $\mathcal{M}, s \models \varphi$, is defined as for $\mathcal{L}_K$, with an additional clause for common knowledge C_B, $B \subseteq A$:

$$\mathcal{M}, s \models C_B\varphi \quad \text{iff} \quad \mathcal{M}, t \models \varphi \text{ for all } t \in S \text{ with } (s,t) \in R_{C_B}.$$

Common Knowledge

Example

$\mathcal{M}, s \models C_{ab}p$
$\mathcal{M}, s \not\models C_{abc}p$



Common Knowledge

Additional axioms and inference rules for common knowledge:

- $C_B(\varphi \rightarrow \psi) \rightarrow (C_B\varphi \rightarrow C_B\psi)$
  (Distribution of C_B over $\rightarrow$)
- $C_B\varphi \rightarrow (\varphi \wedge E_B C_B\varphi)$
  (Mix)
- $C_B(\varphi \rightarrow E_B\varphi) \rightarrow (\varphi \rightarrow C_B\varphi)$
  (Induction of common knowledge)
- From φ, infer $C_B\varphi$
  (Necessitation of C_B)

Theorem

Together with **S5** *axioms and rules, the above axiomatization is sound and complete with respect to epistemic models with common knowledge.* □

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Model Checking

Model Checking

Question 1 (local model checking): Given model $\mathcal{M}$, state s of $\mathcal{M}$, and formula φ. How to test (algorithmically) whether $\mathcal{M}, s \models \varphi$?

Possible answer (Q1): Determine whether $\mathcal{M}, s \models \varphi$ by iteratively unraveling definition of $\models$ relation. For efficiency, cache intermediate results.
This works even if $\mathcal{M}$ is only given implicitly.

Question 2 (global model checking): Given model $\mathcal{M}$ and formula φ. How to determine (algorithmically) the set of all states s of $\mathcal{M}$ such that $\mathcal{M}, s \models \varphi$?

Possible answer (Q2): For all subformulas ψ of φ, determine the sets of states where ψ is true, inductively from small to large subformulas. Details below.

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Definition (Subformula)

Let φ be an $\mathcal{L}_{KC}$ formula. Then the set of subformulas of φ, $subf(\varphi)$, is inductively defined as follows:

$$
\begin{aligned}
subf(p) &= \{p\} \text{ for } p \in P \\
subf(\neg\varphi) &= \{\neg\varphi\} \cup subf(\varphi) \\
subf(\varphi \land \psi) &= \{\varphi \land \psi\} \cup subf(\varphi) \cup subf(\psi) \\
subf(K_a\varphi) &= \{K_a\varphi\} \cup subf(\varphi) \\
subf(C_B\varphi) &= \{C_B\varphi\} \cup subf(\varphi)
\end{aligned}
$$

If $\psi \in subf(\varphi) \setminus \{\varphi\}$, then ψ is called a proper subformula of φ.

Definition

Let a be an agent and $S' \subseteq S$. Then the strong preimage of S' with respect to R_a is the set of states

$$preim_a(S') = \{s \in S \mid s' \in S' \text{ for all } s' \in S \text{ with } (s, s') \in R_a\}.$$

For $B \subseteq A$, we write

$$preim_B(S') = \bigcap_{b \in B} preim_b(S').$$

Notation:

When the model $\mathcal{M}$ and domain S are clear from the context, for a given formula φ, we write $[\![\varphi]\!]$ for the set of states where φ is true, i. e., for $\{s \in S \mid \mathcal{M}, s \models \varphi\}$.

Model Checking

Algorithm

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Let $\mathcal{M} = \langle S, R, V \rangle$ be an (epistemic) Kripke model and $\varphi \in \mathcal{L}_{KC}$ a formula. Let $\varphi_1, \ldots, \varphi_n$ be the subformulas of φ ordered from small to large ($\varphi_n = \varphi$). For $i = 1, \ldots, n$, do:

switch φ_i **do**
    case $p \in P$
        $[\![\varphi_i]\!] := V(p)$
    case $\neg\varphi'$
        $[\![\varphi_i]\!] := S \setminus [\![\varphi']\!]$
    case $\varphi' \wedge \varphi''$
        $[\![\varphi_i]\!] := [\![\varphi']\!] \cap [\![\varphi'']\!]$
    case $K_a\varphi'$
        $[\![\varphi_i]\!] := \mathit{preim}_a([\![\varphi']\!])$
    case $C_B\varphi'$
        $S' := [\![\varphi']\!]$
        while not $\mathit{fixpt}(S')$ **do**
            $S' := S' \cap \mathit{preim}_B(S')$
        end while
        $[\![\varphi_i]\!] := S'$

Example ($[\![\neg K_b(K_a p \wedge q)]\!] = ?$)

$$
\begin{tikzcd}
\boxed{s_1 : p, \neg q} \arrow[r, dash, "a"] \arrow[d, dash, "b"'] & \boxed{s_2 : p, q} \arrow[r, dash, "a"] \arrow[d, dash, "b"'] & \boxed{s_3 : p, q} \arrow[d, dash, "b"] \\
\boxed{s_4 : \neg p, q} \arrow[r, dash, "a"] & \boxed{s_5 : p, q} \arrow[r, dash, "a"] & \boxed{s_6 : p, q}
\end{tikzcd}
$$

$$
\begin{aligned}
[\![p]\!] &= \{s_1, s_2, s_3, s_5, s_6\} \\
[\![q]\!] &= \{s_2, s_3, s_4, s_5, s_6\} \\
[\![K_a p]\!] &= \{s_1, s_2, s_3\} \\
[\![K_a p \wedge q]\!] &= \{s_2, s_3\} \\
[\![K_b(K_a p \wedge q)]\!] &= \emptyset \\
[\![\neg K_b(K_a p \wedge q)]\!] &= \{s_1, s_2, s_3, s_4, s_5, s_6\}
\end{aligned}
$$

Example ($[\![C_{ab}p]\!] = ?$)

$$\boxed{s_1 : p} \overset{a}{—} \boxed{s_2 : p} \overset{a}{—} \boxed{s_3 : p} \overset{a}{—} \boxed{s_4 : p} \overset{b}{—} \boxed{s_5 : p} \overset{c}{—} \boxed{s_6 : p} \overset{a}{—} \boxed{s_7 : p} \overset{b}{—} \boxed{s_8 : \neg p}$$

$$\begin{aligned}
[\![p]\!] &= \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\} \\
S' &:= [\![p]\!] \\
S' &:= S' \cap preim_{ab}(S') = \{s_1, s_2, s_3, s_4, s_5, s_6\} \\
S' &:= S' \cap preim_{ab}(S') = \{s_1, s_2, s_3, s_4, s_5\} \\
S' &:= S' \cap preim_{ab}(S') = \{s_1, s_2, s_3, s_4, s_5\} \quad (\textit{fixpoint!}) \\
[\![C_{ab}p]\!] &= \{s_1, s_2, s_3, s_4, s_5\}
\end{aligned}$$

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Summary

- Basic epistemic language $\mathcal{L}_K$: like propositional logic, plus knowledge modalities
- Kripke semantics: possible worlds, accessibility relations, propositional valuations
- $\mathcal{S}5$ (knowledge): accessibility relations are equivalence relations
- $\mathcal{L}_K$ formulas cannot distinguish between bisimilar models.
- Several axioms have 1-to-1 correspondence to properties of accessibility relations.
- Sound and complete axiomatizations of $\mathcal{K}$ and $\mathcal{S}5$
- Common knowledge = transitive closure of general knowledge ("everybody knows")
- Algorithmic aspect of epistemic logic (so far): model checking