Game Theory

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Lecture Notes

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1 Introduction

1.1 What is Game Theory?

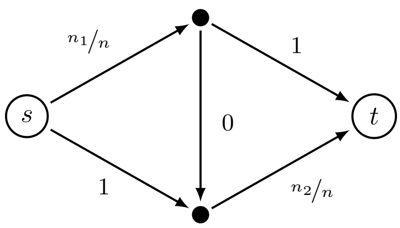
Game Theory is the analysis of strategic decision situations where multiple players interact with each other.

The result of a game depends on the decisions of the involved players and all players are aware of this fact. This raises the question of what the outcome of the game will be assuming each player acts *rationally*, i.e. tries to maximize its own utility, and each player itself assumes that all other players act rationally as well.

Originally, game theory was a research field of theoretical economics but has come to play an increasingly important role in artificial intelligence and computer science, especially when modelling distributed, heterogeneous systems of selfish agents. While solution concepts for games are already known, many algorithmic questions remain to be answered. Since the assumption of players' rationality is more reasonable for artificial agents than for natural agents, game theory might even be more applicable to computer science than to economics.

Example 1 (Beauty contest)**.** Every player chooses a natural number between 1 and 100. The players that come closest to $^2/_3$ of the average win.

Example 2 (Congestion games)**.** Consider a network of streets with travel costs dependent on the number of agents choosing a particular road. The goal of an agent is to travel from s to t minimizing travel cost. Which route will the agents take when there are $n = 2, 3, \dots$ agents? What is the average travel cost per agent?



1.2 Areas of Game Theory

One area of game theory deals with the analysis of **strategic games (normal form games)** where each player selects a strategy from a predefined set of strategies and the outcome of the game is determined by the resulting strategy combination.

Example 3 (Prisoner's dilemma)**.** Two members of a criminal gang are arrested and interrogated separately. The prosecutors lack sufficient evidence to convict the pair on the accused crime. Each prisoner is given the opportunity to *defect* and betray the other prisoner or to *cooperate* with the other by remaining silent. Possible outcomes are:

1. both cooperate and each of them is sentenced to three months of prison,
2. only one prisoner defects and gets out of prison while the cooperating prisoner is sentenced to ten years of prison,
3. both defect and each of them is sentenced to three years of prison.

Another area of game theory regards **extensive-form games** where players perform multiple moves (e.g. chess or repeated strategic games). We differentiate between games with **perfect** information, like chess, and games with **imperfect** information, like poker.

Coalition or negotiation games model situations such as negotiations, distribution of payoffs, or elections.

Mechanism design considers games from the perspective of a game designer who specifies the rules of the game with the intention to optimize the social welfare of all players.

The most important **solution concepts** in game theory are the elimination of dominated strategies, Nash equilibria, and other equilibria notions.

2 Strategic Games

Definition 4 (Strategic game)**.** A **strategic game** $G = \langle N, (A_i), (u_i) \rangle$ consists of

- a finite nonempty **set of players** N,
- for each player $i \in N$, a nonempty set A_i of **actions/strategies** and
- for each player $i \in N$, **payoff function** $u_i : A \to \mathbb{R}$, where

$$A = \prod_{i \in N} A_i.$$

The strategic game G is finite if A is finite.

Instead of payoff functions, we will sometimes use the notion of **preference relations**. The preference relation $\succeq_i$ for player $i \in N$ is defined as follows:

$$a \succeq_i b \text{ iff } u_i(a) \geq u_i(b).$$

We can model a finite strategic game using a **payoff matrix**. As an example, consider a two-player game where each player has two actions. player 1 has actions T and B, player 2 has actions L and R.

		player 2	
		L	R
player 1	T	w_1, w_2	x_1, x_2
	B	y_1, y_2	z_1, z_2

If player 1 chooses action T and player 2 chooses action L, then player 1 gets payoff w_1 and player 2 gets payoff w_2, i. e. in every cell of the payoff matrix the first value is the payoff of player 1, and the second is the payoff of player 2.

Example 5 (Prisoner's dilemma)**.** Consider the prisoner's dilemma as described above. Let D be the strategy to defect and C to cooperate with the fellow prisoner. The payoffs correspond to the months in prison:

	C	D
C	$-3, -3$	$-120, \quad 0$
D	$0, -120$	$-36, -36$

Example 6 (Hawk and dove)**.** In a fight for resources, two players can behave either like a hawk (H) or like a dove (D). If both players behave like doves, both players share the benefit, if a hawk meets a dove, the hawk wins and gets the bigger part. However if two hawks meet each other, then the benefit gets lost completely because they will fight each other.

	D	H
D	$3, 3$	$1, 4$
H	$4, 1$	$0, 0$

Example 7 (Matching pennies)**.** Two players can choose either heads (H) or tails (T) of a coin. If both players make the same choice, then player 1 receives one Euro from player 1, and if they make different choices, then player 2 gets one Euro from player 1.

	H	T
H	$1, -1$	$-1, \ \ 1$
T	$-1, \ \ 1$	$1, -1$

Example 8 (Bach or Stravinsky)**.** Two persons, one of whom prefers Bach whereas the other one prefers Strawinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite concert. Let B be the action of going to the Bach concert, S the action of going to the Strawinsky concert.

		Strawinsky enthusiast	
		B	S
Bach enthusiast	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

2.1 Dominated Strategies

Notation 9. Let $a = (a_i)_{i \in N}$ be **strategy profile** ($a \in A = \prod_{i \in N} A_i$). Furthermore we define $a_{-i} := (a_j)_{j \in N \setminus \{i\}}$ and $(a_{-i}, a_i) = (a_j)_{j \in (N \setminus \{i\}) \cup \{i\}} = (a_j)_{j \in N} = a$.

Definition 10 (Strictly dominated strategy)**.** An action $a_i \in A_i$ in strategic game $\langle N, (A_i), (u_i) \rangle$ is called strictly dominated if there is an action $a_i^+ \in A_i$ such that for all $a \in A$:

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

Remark 11. It is never rational to play strictly dominated strategies.

2.1.1 Iterative Elimination of Strictly Dominated Strategies

- cancel out strategies that are strictly dominated as long as possible.
- if there is only one strategy profile left then that is the solution.

Example 12. Firstly, strike out the first row which is strictly dominated by the second row. Then eliminate the second column which is strictly dominated by the first column.

	C	D
C	~~3,3~~	~~0,4~~
D	4,0	1,1

	C	D
D	~~4,0~~	1,1

Example 13. Iterative elimination of strictly dominated strategies in three steps:

	L	R
T	2,1	0,0
M	1,2	2,1
B	~~0,0~~	~~1,1~~

	L	R
T	2,1	~~0,0~~
M	1,2	~~2,1~~

	L
T	2,1
M	~~1,2~~

Remark 14. Only rarely there is strict dominances between actions.

Remark 15. The result of iterative elimination of strict dominance is unique, i. e., independent of the elimination order.

Definition 16 (Weakly dominated strategies)**.** An action $a_i \in A_i$ in game $\langle N, (A_i), (u_i) \rangle$ is called weakly dominated, if there is an action $a_i^+ \in A_i$ such that for all $a \in A$

$$u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+)$$

and for at least one $a \in A$, we have

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

Example 17. The result of iterative elimination of weakly dominated strategies is in general not unique and depends on the elimination order. To see this, consider the following game:

	L	R
T	$2, 1$	$0, 0$
M	$2, 1$	$1, 1$
B	$0, 0$	$1, 1$

First of all, we eliminate action T ($T \leq M$), then we eliminate action L ($L \leq R$). As a result every possible strategy profile, i.e. MR or BR, in the reduced game has payoff profile $(1, 1)$. An alternative to this elimination order is to first eliminate B ($B \leq M$), and then R ($R \leq L$) leading to a reduced game with strategy profiles TL and ML. Both strategy profiles have payoff profiles $(2, 1)$ and give player 1 –unlike the other ordering–a higher utility compared to the reduced game where only strategy profiles MR and BR remained.

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