10. Repeated Games

- If a strategic game is played repeatedly, then the players might behave differently than in the one-shot setting.

a) finitely repeated: you play the game for a known number of rounds.

b) infinitely repeated: infinite number of rounds.

c) indefinitely repeated: you have a given probability p that the current round is the last one.

<u>Def.:</u> Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game (the "stage game"). Let $A = \prod_{i=1}^{n} A_i$. Then a <u>repeated game</u> with $k \in \mathbb{N} \cup \{\infty\}$ moves is an extensive game with simultaneous moves $\Gamma = \langle N, A, H, P, (v_i) \rangle$ with

- $H = \{\langle\rangle\} \cup \bigcup_{t=1}^{k} A^t$
- $P(h) = N$ for all nontern. hist. h
- If $k \in \mathbb{N}$: $v_i(h) = \sum_{t=1}^{k} u_i(a^t)$ for $h = a^1 a^2 \dots a^k$ (for term. histories h)

If $k = \infty$: Let $\delta \in (0,1)$ be a discount factor.

Then $v_i(h) = \sum_{t=1}^{\infty} \delta^{t-1} \cdot u_i(a^t)$

$h = a^1 a^2 \dots a^t \dots$

<u>Example:</u> $\delta = 1/2$. $u_i(a^t) = 1$

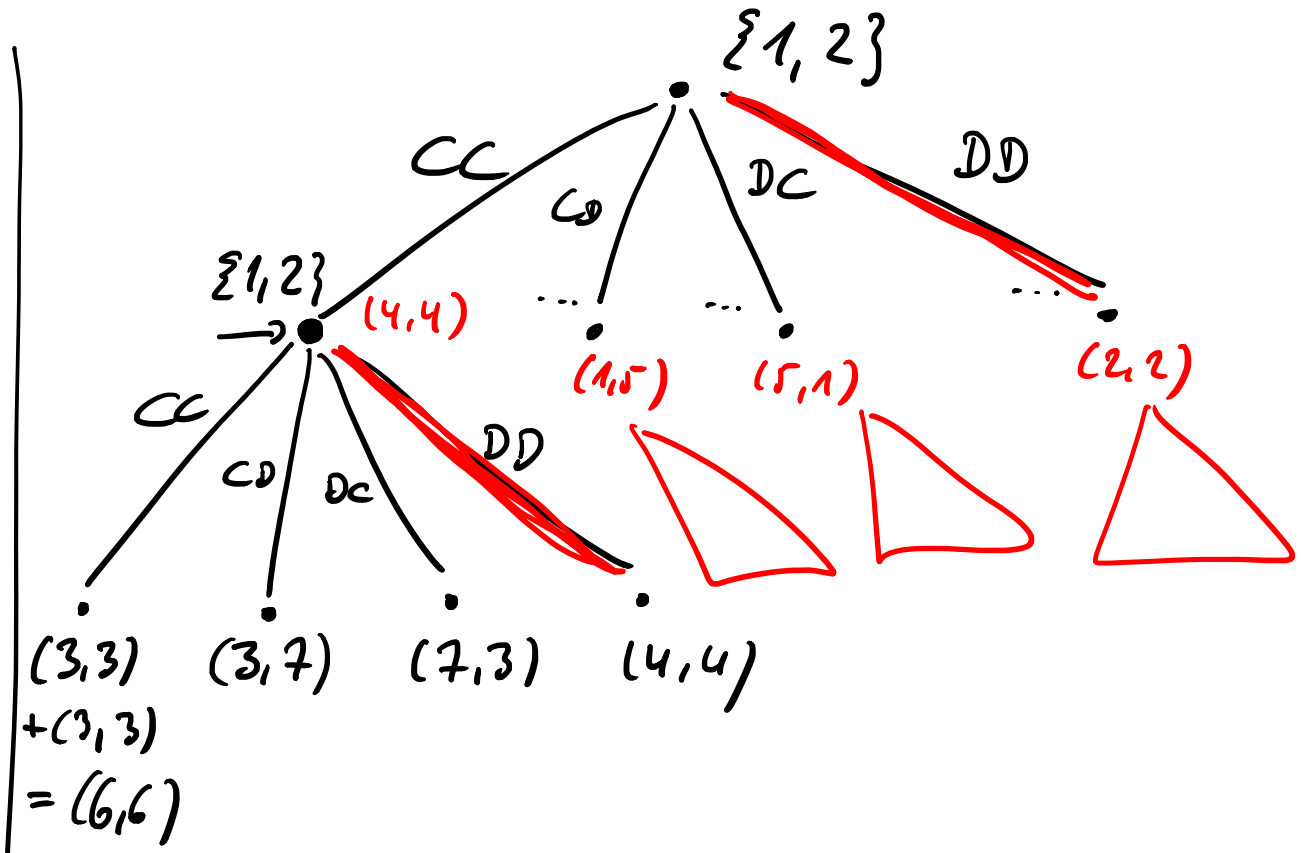
$$v_i(h) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t) = 1 \cdot 1 + 1/2 \cdot 1 + 1/4 \cdot 1 + \dots$$

$$= 1 + 1/2 + 1/4 + 1/8 + \dots = 2$$

Example: Finitely repeated PD with $k=2$.

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

	C	D
C	6, 6	3, 7
D	7, 3	4, 4



- No matter how many rounds $k \in \mathbb{N}$ are played, DD is always the unique subgame-perfect equilibrium.

Infinitely repeated games

Discounting: What is $1 + \delta + \delta^2 + \delta^3 + \dots$?

It converges to $\frac{1}{1-\delta}$ for $0 < \delta < 1$.

Proof:

$$
\begin{aligned}
x &= 1 + \delta + \delta^2 + \delta^3 + \dots \\
&= 1 + \delta \underbrace{(1 + \delta + \delta^2 + \dots)}_{x} \\
&= 1 + \delta x
\end{aligned}
$$

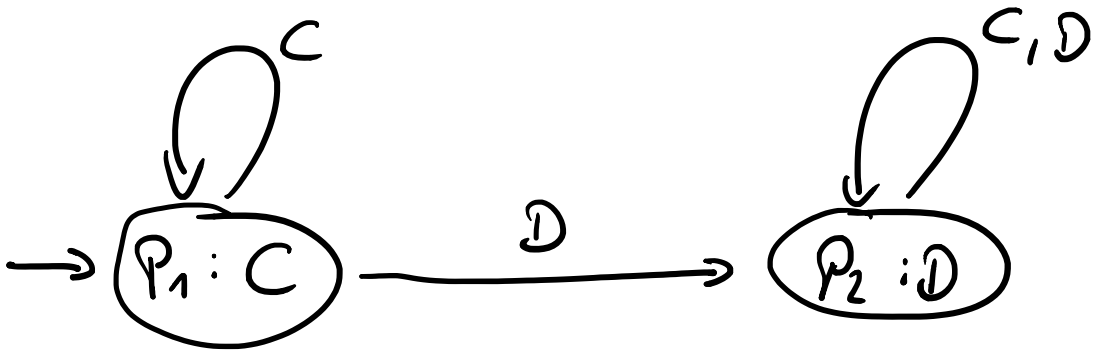
$$\Rightarrow \quad x - \delta x = 1$$

$$\Leftrightarrow \quad x(1-\delta) = 1 \quad \Leftrightarrow \quad x = \frac{1}{1-\delta}$$

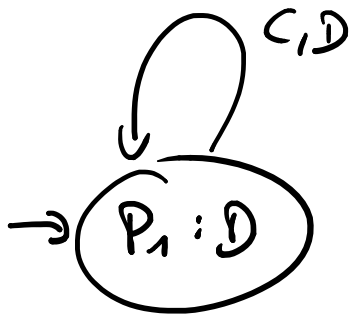
Strategies for infinitely repeated games:

Finite automata (Moore automata):

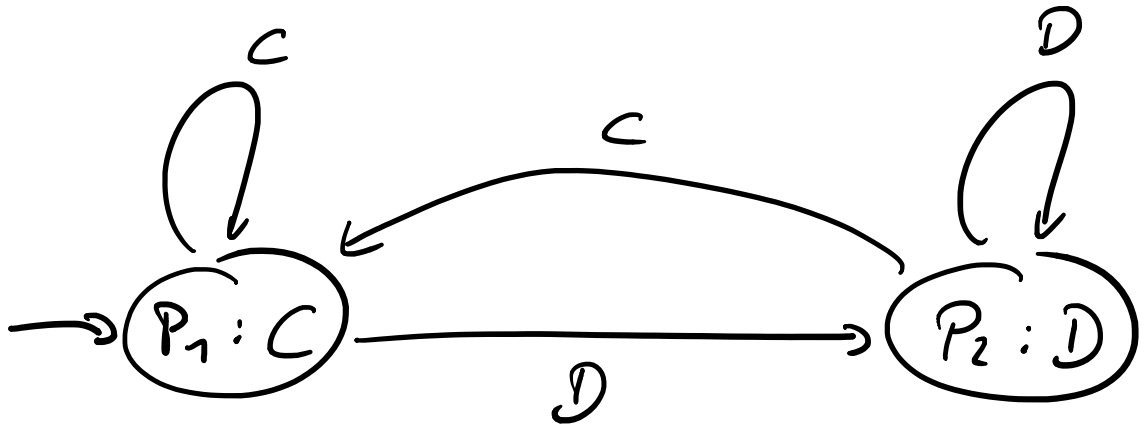
Example:



grim strategy, g



defecting strategy, d



tit-for-tat strategy, t

Is (g,g) an equilibrium?

$$V_i^{\delta}(g,g) = 3 + \delta \cdot 3 + \delta^2 \cdot 3 + \dots = 3 \cdot (1 + \delta + \delta^2 + \dots) = 3 \cdot \frac{1}{1-\delta}$$

Then, the unique run the players get is $(C,C), (C,C), (C,C), \dots$

Is (g, g) a Nash equilibrium?

Only reasonable candidate(s) for better responses to g:

g' : choose D at some point <u>unprovoked</u>, and then ever after

$$\Rightarrow \quad O(g', g) = \langle (C,C), (C,C), \ldots, (C,C), (D,C), (D,D), \ldots, (D,D), \ldots \rangle$$

W.L.O.G, assume that the first unprovoked Defection in g' happens in the first step.

$$\Longrightarrow \quad g' = d.$$

So: Determine $v_1(o(d,g))$, compare to $v_1(o(g,g))$.

We already know $v_1(o(g,g)) = \boxed{3 \cdot \frac{1}{1-\delta}}$.

$$
\begin{aligned}
v_1(o(d,g)) &= v_1(\langle (D,C), (D,D), (D,D), \dots \rangle) \\
&= 4 + \delta \cdot 1 + \delta^2 \cdot 1 + \dots \\
&= 4 + \delta (1 + \delta + \delta^2 + \dots) \\
&= 4 + \delta \cdot \frac{1}{1-\delta} = \boxed{4 + \frac{\delta}{1-\delta}}
\end{aligned}
$$

(g,g) is a NE iff $3 \cdot \frac{1}{1-\delta} \geq 4 + \frac{\delta}{1-\delta}$

$$
\begin{aligned}
&\Leftrightarrow \quad 3 \geq 4(1-\delta) + \delta = 4 - 4\delta + \delta = 4 - 3\delta \\
&\Leftrightarrow \quad 3\delta \geq 1 \quad \Leftrightarrow \quad \delta \geq 1/3
\end{aligned}
$$

This means: (g, g) is a NE if $\delta \geq 1/3$ is large enough.

(then, g is at least as good a response to g as d is.)

- Also, (d, d) is a NE!
- Also, (t, t) is a NE (for $\delta \geq 1/3$).

<u>Positive message:</u> In repeated games, there are other NEs than just (D, D).

<u>Negative message:</u> Which NE to play?

Indefinitely repeated games

Theory <u>very</u> similar to theory for infinitely repeated games, because:

discount factor $\approx$ probability that there is a next round. $\square$