10. Repeated Games

- If a strategic game is played repeatedly, then the players might behave differently than in the one-shot setting.

a) finitely repeated: you play the game for a known number of rounds.

b) infinitely repeated: infinite number of rounds.

c) indefinitely repeated: you have a given probability p that the current round is the last one.

Def.: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game (the "stage game"). Let $A = \prod_{i=1}^{n} A_i$. Then a <u>repeated game</u> with $k \in \mathbb{N} \cup \{\infty\}$ moves is an extensive game with simultaneous moves $\Gamma = \langle N, A, H, P, (v_i) \rangle$ with

- $H = \{\langle\rangle\} \cup \bigcup_{t=1}^{k} A^t$
- $P(h) = N$ for all nontrm. hist. h
- If $k \in \mathbb{N}$: $v_i(h) = \sum_{t=1}^{k} u_i(a^t)$ for $h = a^1 a^2 \dots a^k$ (for trm. histories h)

If $k = \infty$: Let $\delta \in (0,1)$ be a discount factor.

Then $v_i(h) = \sum_{t=1}^{\infty} \delta^{t-1} \cdot u_i(a^t)$

$h = a^1 a^2 \dots a^t \dots$

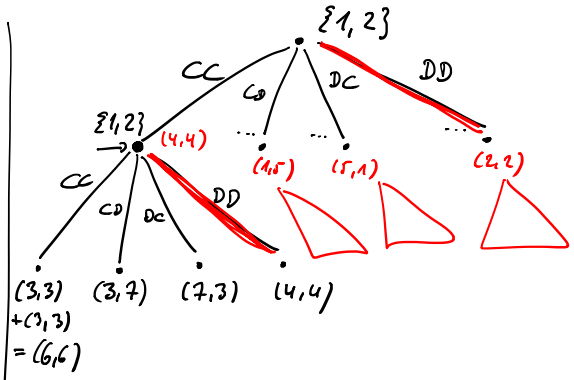
<u>Example:</u> $\delta = 1/2$. $u_i(a^t) = 1$

$$v_i(h) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t) = 1 \cdot 1 + 1/2 \cdot 1 + 1/4 \cdot 1 + \dots$$
$$= 1 + 1/2 + 1/4 + 1/8 + \dots = 2$$

<u>Example:</u> Finitely repeated PD with $k = 2$.

	C	D
C	3,3	0,4
D	4,0	1,1

	C	D
C	6,6	3,7
D	7,3	4,4



- No matter how many rounds $k \in \mathbb{N}$ are played, DD is always the unique subgame-perfect equilibrium.

Infinitely repeated games

Discounting: What is $1+\delta+\delta^2+\delta^3+\dots$?

It converges to $\frac{1}{1-\delta}$ for $0<\delta<1$.

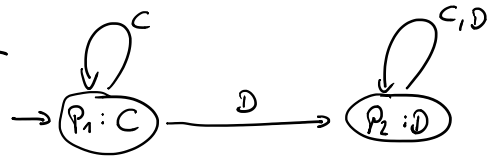
Proof:

$$x = 1+\delta+\delta^2+\delta^3+\dots$$
$$= 1+\delta\underbrace{(1+\delta+\delta^2+\dots)}_{x}$$
$$= 1+\delta x$$
$$\Rightarrow x-\delta x = 1$$
$$\Leftrightarrow x(1-\delta)=1 \quad \Leftrightarrow x=\frac{1}{1-\delta}$$

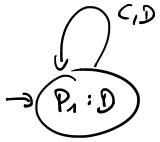
Strategies for infinitely repeated games:

Finite automata (Moore automata):

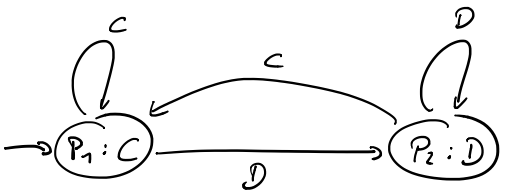
Example:



grim strategy, g



defecting strategy, d



tit-for-tat strategy, t

Is (g,g) an equilibrium?

$$V_1^0(g,g) = 3+\delta\cdot 3+\delta^2\cdot 3+\dots = 3\cdot(1+\delta+\delta^2+\dots) = 3\cdot\frac{1}{1-\delta}$$

Then, the unique run the players get is $(C,C), (C,C), (C,C), \dots$

Is (g,g) a Nash equilibrium?

Only reasonable candidate(s) for better responses to g:

g': choose D at some point unprovoked, and then ever after

$$\Rightarrow O(g',g) = \langle (C,C), (C,C), \dots, (C,C), (D,C), (D,D), \dots, (D,D), \dots \rangle$$

W.L.o.G, assume that the first unprovoked Defection in g' happens in the first step.

$$\Rightarrow g' = d.$$

So: Determine $v_1(o(d,g))$, compare to $v_1(o(g,g))$.

We already know $v_1(o(g,g)) = \boxed{3 \cdot \frac{1}{1-\delta}}$.

$$
\begin{aligned}
v_1(o(d,g)) &= v_1(\langle (D,C), (D,D), (D,D), \dots \rangle) \\
&= 4 + \delta \cdot 1 + \delta^2 \cdot 1 + \dots \\
&= 4 + \delta\,(1 + \delta + \delta^2 + \dots) \\
&= 4 + \delta \cdot \frac{1}{1-\delta} = \boxed{4 + \frac{\delta}{1-\delta}}
\end{aligned}
$$

(g,g) is a NE iff $3 \cdot \frac{1}{1-\delta} \geq 4 + \frac{\delta}{1-\delta}$

$$\iff 3 \geq 4(1-\delta) + \delta = 4 - 4\delta + \delta = 4 - 3\delta$$

$$\iff 3\delta \geq 1 \iff \delta \geq 1/3$$

This means: (g,g) is a NE if $\delta \geq 1/3$ is large enough.
(then, g is at least as good a response to g as d is.)

- Also, (d,d) is a NE!
- Also, (t,t) is a NE (for $\delta \geq 1/3$).

<u>Positive message:</u> In repeated games, there are other NEs than just (D,D).

<u>Negative message:</u> Which NE to play?

<u>Indefinitely repeated games</u>

Theory <u>very</u> similar to theory for infinitely repeated games, because:

discount factor $\approx$ probability that there is a next round. $\square$