Introduction to Modal Logic

B. Nebel, S. Wölfl | University of Freiburg
Summer 2015 | Department of Computer Science

Exercise Sheet 5

Due: 03-06-2015

The exercise sheets may be worked on, and handed in, in groups of two students. In that case please indicate both names on your solution. The solution must be handed in on Wednesday *before* the lecture (either on paper or electronically by email to `woelfl@informatik.uni-freiburg.de`).

Exercise 5.1 (4 points)

We consider the fragment of regular PDL in which program terms may only be formed by using the choice and the composition constructor, but not the iteration constructor. Define a function $\mathrm{ST}_x(\cdot)$ that translates each PDL formula φ in this fragment into a FO formula $\mathrm{ST}_x(\varphi)$ such that:

$$\mathcal{M}^{\mathcal{S}} \models_s \varphi \iff \mathcal{S}, (x \mapsto s) \models \mathrm{ST}_x(\varphi).$$

In this equivalence, $\mathcal{S}$ is a first-order relational structure with a binary relation for each basic PDL program (and of course a unary relation for each propostional variable), while $\mathcal{M}^{\mathcal{S}}$ is the Kripke model obtained from $\mathcal{S}$ by adding binary relations for complex program terms such that for all π_1 and π_2, $R_{\pi_1 \cup \pi_2} = R_{\pi_1} \cup R_{\pi_2}$ and $R_{\pi_1;\pi_2} = R_{\pi_1} \circ R_{\pi_2}$.

Hint: You will need a translation $\mathrm{ST}_{x,y}(\pi)$ that maps each program term π to a FO formula in which the variables x and y are the only free variables.

Exercise 5.2 (3 points)

We consider the basic modal language $\mathcal{L}$ with a single modal operator $\Diamond$.
Let F be a class of Kripke frames and $\mathsf{M}(\mathsf{F})$ be the class of Kripke models defined on some frame in F. Furthermore, let $\Sigma \cup \{\varphi\}$ be a set of $\mathcal{L}$-formulae. Consider now the following claims:

(i) For each model $\mathcal{M}$ in $\mathsf{M}(\mathsf{F})$, if $\mathcal{M} \models \Sigma$, then $\mathcal{M} \models \varphi$.

(ii) For each frame $\mathcal{F}$ in F, if $\mathcal{F} \models \Sigma$, then $\mathcal{F} \models \varphi$.

Prove or disprove the following claims:

(a) (i) $\Rightarrow$ (ii).

(b) (ii) $\Rightarrow$ (i).

Exercise 5.3 (3 points)

Consider the modal similarity type τ with a nullary modal operator $\bigcirc_r$ for each non-negative real number $r \in \mathbb{R}_0^+$ and a single unary modal operator $\Diamond$.
Show that there exists a set Σ of $\mathcal{L}$-formulae that is satisfiable, but not satisfiable in any model with an at most countably infinite number of states.

Hint: Consider a model $\mathcal{M}$ with $\mathcal{M} \models_s \bigcirc_r$ if and only if $r = s, \ldots$