Introduction to Modal Logic

B. Nebel, S. Wölfl
Summer 2015

University of Freiburg
Department of Computer Science

Exercise Sheet 2

Due: 06-05-2015

The exercise sheets may be worked on, and handed in, in groups of two students. In that case please indicate both names on your solution. The solution must be handed in on Wednesday *before* the lecture (either on paper or electronically by email to `woelfl@informatik.uni-freiburg.de`).

Exercise 2.1 (3 points)

Show that a transitive relation R on a finite set S is well-founded if and only if the relation is irreflexive.

Exercise 2.2 (1 + 2 points)

Let $P = \{\mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{s}\}$ be a set of propositional variables. Consider the language $\mathcal{L} = \mathcal{L}(P)$ with formulae defined as follows:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi_1 \vee \varphi_2) \mid \langle i \rangle \varphi \quad (i = 0, \ldots, 9).$$

(1) Informally, a *subformula* of an $\mathcal{L}$-formula φ is any $\mathcal{L}$-formula ψ that forms a substring of φ, i.e., it has been used to form φ according to the given rules. Also φ counts as a subformula of itself. Provide a recursive definition of the operator $\mathrm{Sub}(\cdot)$ that assigns to each $\mathcal{L}$-formula the set of its subformulae.

(2) The length of an $\mathcal{L}$-formula φ is the number of symbols used to write φ. For example, the length of the formula $\varphi = \langle 1 \rangle \langle 2 \rangle (\mathrm{p} \vee \langle 1 \rangle \mathrm{p})$ is $|\varphi| = 14$ (notice that we count all parentheses). Show that $|\mathrm{Sub}(\varphi)| \leq |\varphi|$.

Exercise 2.3 (4 points)

Use the procedure presented in section 1.2 of the lecture notes to check whether the following formulae are valid with respect to valuation semantics. If the formula is not valid, provide a counter-example (i.e., a model in which the formula is not valid).

(a) $\Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$

(b) $\Diamond(p \wedge \neg q) \to (p \to \Box(p \vee \Box\Diamond q))$

(c) $\Diamond\Box p \to (\Diamond q \to \Box(p \wedge \Diamond q))$