Epistemic Logics

Muddy children puzzle

The story ...

Five children are playing together. The children have been told that they should not get dirty (otherwise, . . .). Nevertheless, three of them get dirty: they have mud on their foreheads. Each child can see the mud on the heads of the other children, but no child can see the mud on its own head. When the father of the children shows up, he says to the children: *"At least one of you has mud on the forehead."* After that the father asks several times: *"Does any of you know whether you have mud on your forehead?"* Each time the children answer truthfully and simultaneously, but they can hear the answers of the other children.

What happens?

Your solution?

An inductive argument

Lemma

If n children are playing together and k of them get muddy, then the first $k-1$ times when the father asks, all muddy children will answer "No". Afterwards all muddy children will answer "Yes".

Proof (sort of . . .)

The situation before the father approaches:
When k children are muddy, each muddy child sees $k-1$ muddy children and hence knows that $k-1$ or k children are muddy.

After the initial statement of the father:
Each child knows that one of them is muddy, but also knows that all other children know that one of them is muddy.

An inductive argument II

Proof (. . .)

Now the inductive argument:

$k = 1$: Clearly, the muddy child will answer "Yes".

$k = 2$: Let x and y be the muddy children. After the first round, x and y do not know whether they are muddy or not. After the 2nd question of the father, x may reason as follows: If just one of us is muddy, the muddy one could have discovered that already after the 1st question. Hence at least two of us must be muddy. Since I can see only one muddy sister (namely y), I must be muddy as well.
Of course, y will reason in the same way, and thus both answer "Yes".

$k = 3$: Let x, y, and z be the muddy children. z knows 3 or 2 children are muddy. If he were not muddy, only two children (namely x and y) are muddy. Hence, they would have answered "Yes" after the 2nd round. As they did not, z can infer, that he must be muddy as well.

. . . □

A formal approach

Agent frame : a Kripke frame $\mathcal{F} = \langle S, \{R_a\}_{a \in A} \rangle$
$a \in A$ are referred to as **agents**

Epistemic principles

1. $K_a(p \rightarrow q) \rightarrow (K_a p \rightarrow K_a q)$
2. $K_a p \rightarrow p$ (Knowledge Axiom)
3. $K_a p \rightarrow K_a K_a p$ (Positive Introspection)
4. $\neg K_a p \rightarrow K_a \neg K_a p$ (Negative Introspection)

- Knowledge an **S5** modality?

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- Knowledge an **S5** modality?

Enriching the language

Let $G \subseteq A$ be a set of agents.

$E_G\, \varphi$: Everybody in the group G knows that φ

$C_G\, \varphi$: It is common knowledge in the group G that φ

$D_G\, \varphi$: It is distributed knowledge in the group G that φ

Everybody knows ...

Everybody knows:

$$\mathcal{M} \models_s \mathrm{E}_G\varphi \iff \mathcal{M} \models_s \mathrm{K}_a\varphi, \text{ for each } a \in G$$

Distributed knowledge:

$$\mathcal{M} \models_s \mathrm{D}_G\varphi \iff \mathcal{M} \models_t \varphi, \text{ for each } t \in \bigcap_{a\in G} sR_a$$

Define: $\mathrm{E}^0_G\varphi := \varphi$, $\mathrm{E}^k_G\varphi := \mathrm{E}_G\mathrm{E}^{k-1}_G\varphi$.

Common knowledge:

$$\mathcal{M} \models_s \mathrm{C}_G\varphi \iff \mathcal{M} \models \mathrm{E}^k_G\varphi \text{ for each } k \geq 1$$

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Define: $\mathrm{E}_G^0 \varphi := \varphi$, $\mathrm{E}_G^k \varphi := \mathrm{E}_G \mathrm{E}_G^{k-1} \varphi$.

Common knowledge:

$$\mathcal{M} \models_s \mathrm{C}_G \varphi \iff \mathcal{M} \models \mathrm{E}_G^k \varphi \text{ for each } k \geq 1$$

Reachability

Let $\mathcal{F}$ be an agent frame, $G \subseteq A$.

G-path : a sequence of states $s_0, \ldots, s_n$ such that each (s_i, s_{i+1}) is contained in R_a for some $a \in G$

G-reachable : a state s' is G-reachable from state s if there exists a G-path from s to s'.

Fact

- *$\mathcal{M} \models_s \mathrm{E}_G^k \varphi$ if and only if for each s' that is G-reachable from s in k steps, $\mathcal{M} \models_{s'} \varphi$.*
- *$\mathcal{M} \models_s \mathrm{C}_G \varphi$ if and only if for each s' that is G-reachable from s, $\mathcal{M} \models_{s'} \varphi$.*

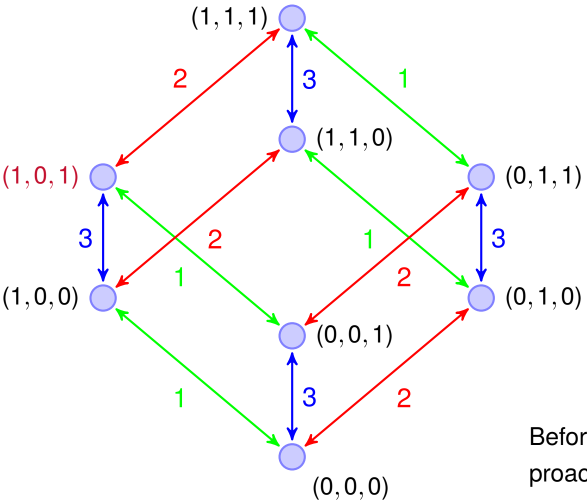
Axiomatic characterization

- C_G and D_G are **S5** modalities (given that all K_a are **S5** modalities)
- $E_G\varphi \leftrightarrow \bigwedge_{a\in G} K_a\varphi$, where G is finite
- $C_G\varphi \leftrightarrow E_G(\varphi \wedge C_G\varphi)$
- $D_{\{a\}}\varphi \leftrightarrow K_a\varphi$
- $D_G\varphi \rightarrow D_{G'}\varphi$, where $G \subseteq G'$
- $C_{G'}\varphi \rightarrow C_G\varphi$, where $G \subseteq G'$

Furthermore,

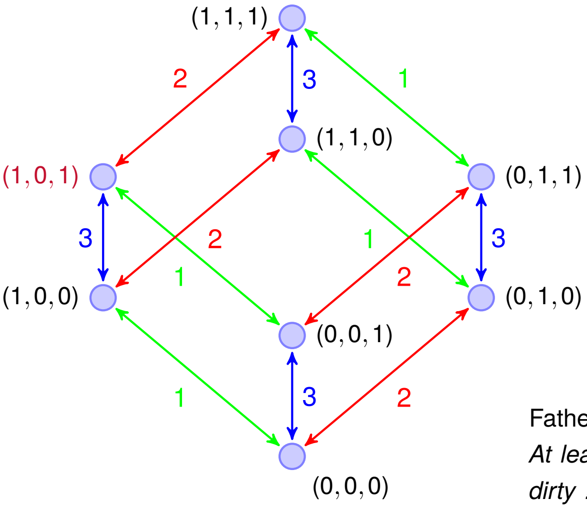
- If $\varphi \rightarrow E_G(\psi \wedge \varphi)$ is valid in a model $\mathcal{M}$, then so is $\varphi \rightarrow C_G\psi$ (Induction Rule)

Muddy Children Puzzle



Before the father approaches

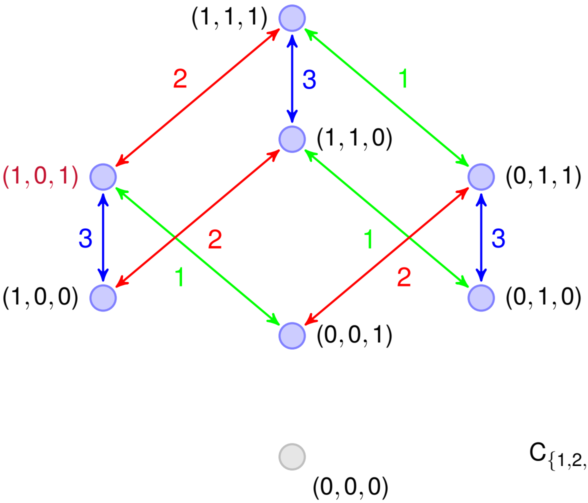
Muddy Children Puzzle



Father:

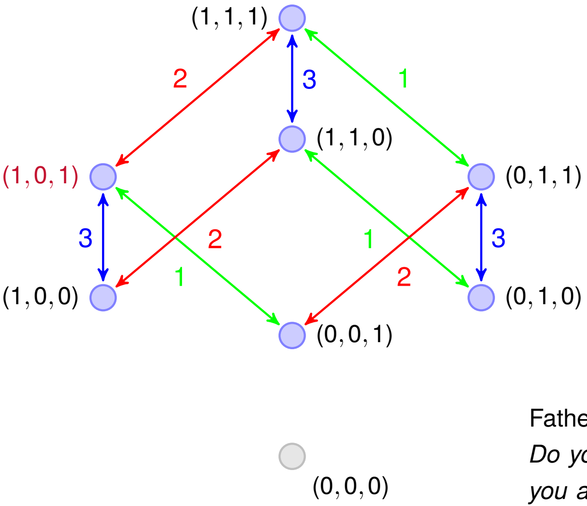
At least one of you is dirty !

Muddy Children Puzzle



$C_{\{1,2,3\}}(p_1 \vee p_2 \vee p_3)$

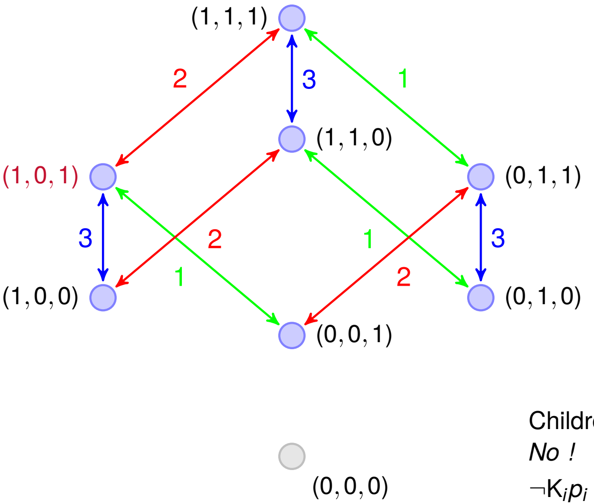
Muddy Children Puzzle



Father:

Do you know whether you are dirty ?

Muddy Children Puzzle

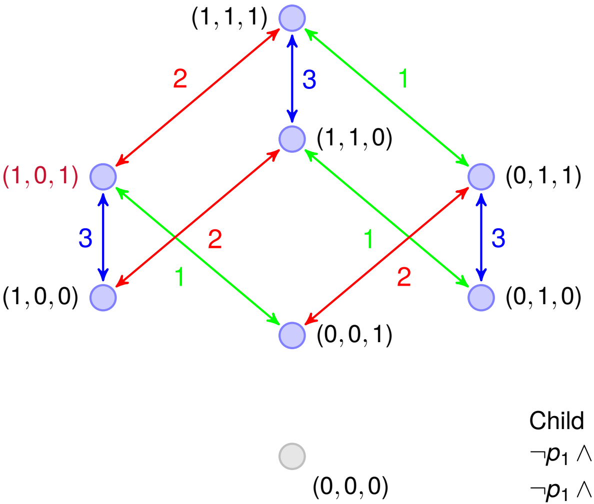


Children:

No !

$\neg K_i p_i \wedge \neg K_i \neg p_i$

Muddy Children Puzzle

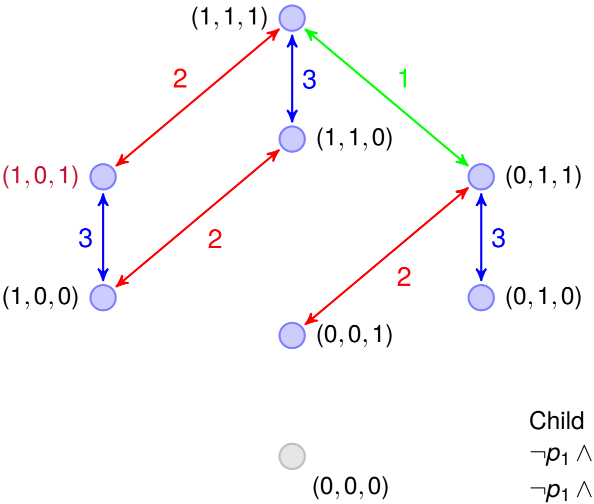


Child 1 reasons:

$\neg p_1 \wedge \neg p_2 \wedge p_3 \rightarrow K_3 p_3$

$\neg p_1 \wedge p_2 \wedge \neg p_3 \rightarrow K_2 p_2$

Muddy Children Puzzle



Child 1 reasons:

$\neg p_1 \land \neg p_2 \land p_3 \rightarrow K_3 p_3$

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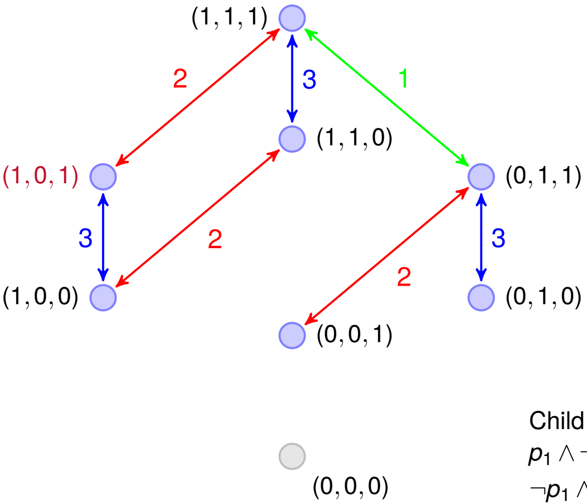
Muddy Children Puzzle

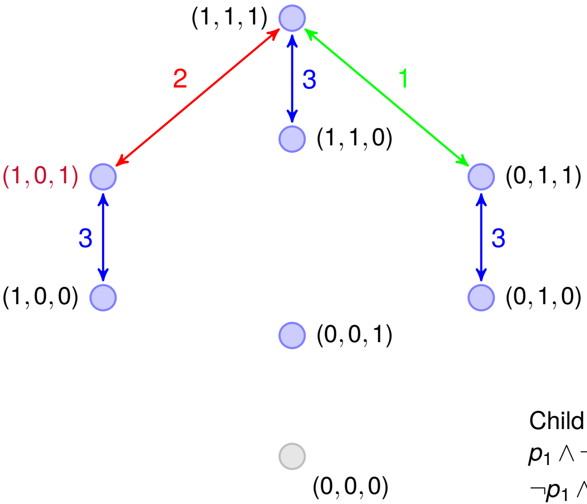


Child 2 reasons:

$p_1 \wedge \neg p_2 \wedge \neg p_3 \rightarrow K_1 p_1$

$\neg p_1 \wedge \neg p_2 \wedge p_3 \rightarrow K_3 p_3$

Muddy Children Puzzle

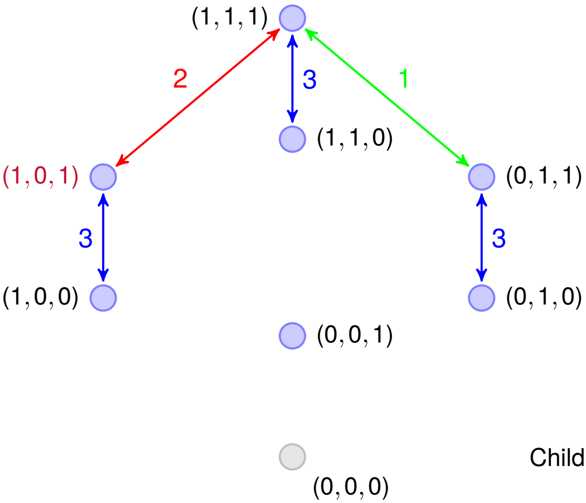


Child 2 reasons:

$p_1 \wedge \neg p_2 \wedge \neg p_3 \rightarrow K_1 p_1$

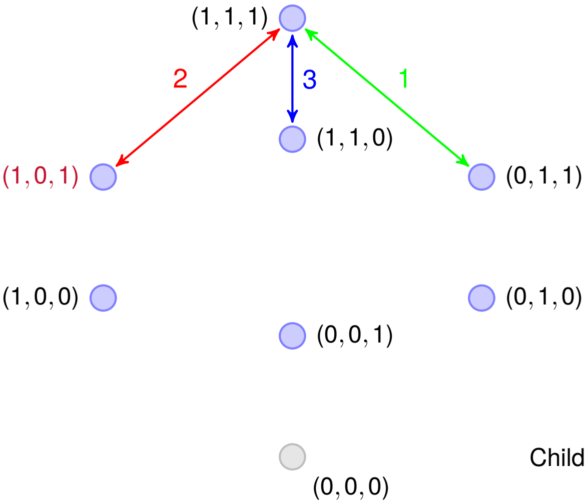
$\neg p_1 \wedge \neg p_2 \wedge p_3 \rightarrow K_3 p_3$

Muddy Children Puzzle



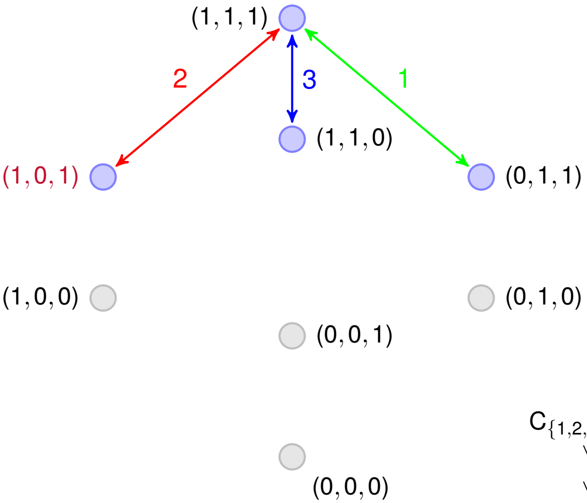
Child 3 ...

Muddy Children Puzzle



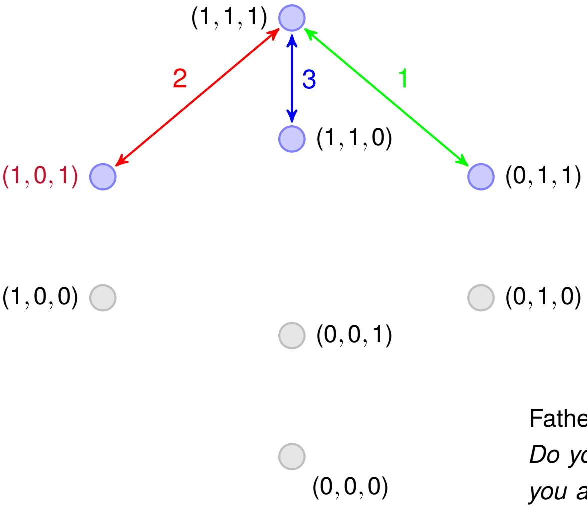
Child 3 ...

Muddy Children Puzzle



$$C_{\{1,2,3\}}(p_1 \wedge p_2 \vee p_1 \wedge p_3 \vee p_2 \wedge p_3)$$

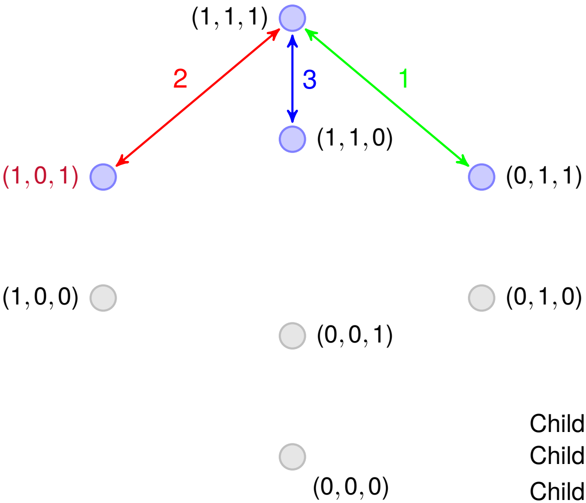
Muddy Children Puzzle



Father:

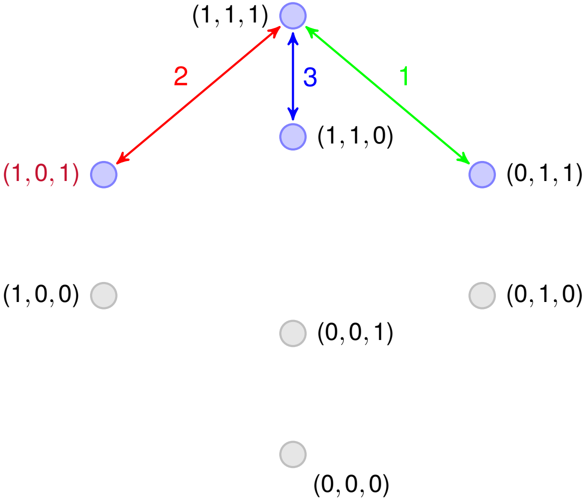
Do you know whether you are dirty ?

Muddy Children Puzzle



Child 1: *Yes !*
Child 2: *No !*
Child 3: *Yes !*

Muddy Children Puzzle



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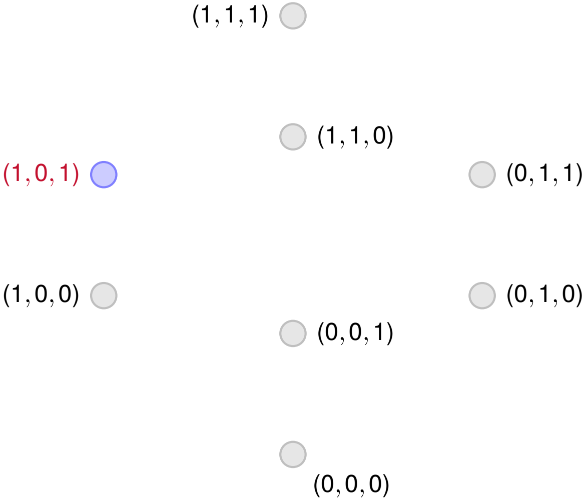

Literature

Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi
Reasoning about Knowledge
MIT Press, 1995