Can we transfer the impossibility result to social choice function?

Yes!

It is even worse:

→ One can show that it is impossible to design social choice functions that immune to "strategic manipulation."



Def.

A social choice function $f$ can be strategically manipulated by voter $i$ if there a preferences $\prec_1, \prec_2, \dots, \prec_i, \dots, \prec_n, \prec_i' \in L$ such that $\boxed{a \prec_i b}$ for $a = f(\prec_1, \dots, \prec_i, \dots, \prec_n)$ and $b = f(\prec_1, \dots, \prec_i', \dots, \prec_n)$.

The function $f$ is incentive compatible if $f$ cannot be manipulated.



Def

A social choice function is monotone if $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$, $f(\prec_1, \dots, \prec_i', \dots, \prec_n) = b$ and $a \neq b$ implies $b \prec_i a$ and $a \prec_i' b$.

Proposition

A social choice function is monotone iff it is incentive compatible.

Proof:

$\Rightarrow$ Let $f$ be monotone: if $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$ and $f(\prec_1, \dots, \prec_i', \dots, \prec_n) = b$ and $a \neq b$ then $b \prec_i a$ and $a \prec_i' b$. Then it cannot be that there exist $\prec_1, \dots, \prec_i, \dots, \prec_n, \prec_i'$ with $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$



and $f(\prec_1, \dots, \prec_i', \dots, \prec_n) \neq b$ and $a \prec_i b$.

$\Leftarrow$: Suppose monotonicity is violated, i.e., there are $a, b \in A$: $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$ and $f(\prec_1, \dots, \prec_i', \dots, \prec_n) = b$ and (1) $a \prec_i b$ or (2) $b \prec_i' a$. If (1), assuming $\prec_i$ is the true preference of $i$, $i$ could declare $\prec_i'$ in order to get the more preferred $b$ as a result. If (2), then assume that $\prec_i'$ is the true preference and $i$ can manipulate again. $\square$

**Def**

A function $f: X \to Y$ is called <u>surjective</u> or <u>onto</u> if for all $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

**Def** (Dictatorship)

Voter $i$ is called a <u>dictator</u> in a social choice function $f$ if for all $<_1, \dots, <_i, \dots, <_n \in L$, $f(<_1, \dots, <_i, \dots, <_n) = a$, where $a$ is the unique alternative with $b <_i a$ for all $b \in A$, $b \neq a$. Function $f$ is a <u>dictatorship</u> if there is a dictator in $f$.



**Theorem** (Gibbard-Satterthwaite)

If $f$ is an <u>incentive compatible</u> and <u>onto</u> social function over <u>at least three alternatives</u>, then $f$ is a <u>dictatorship</u>.

**Proof idea:**

- use Arrow's theorem
- construct a social welfare function from a social choice function.
- construction is done starting from an incentive compatible and onto and non-dictatorship social choice function
- the construction ensures that the social welfare function satisfies IIA, unanimity, and non-dictatorship.



**Notation:**

Let $S \subseteq A$ and $< \in L$. Then $<^S$ denotes the order obtained from $<$ by moving all elements from $S$ to the top, without changing the order between elements in $S$ and $A \setminus S$:

Example

$<$

c
a
b
d

$<^{\{a,d\}}$

a
d
c
b