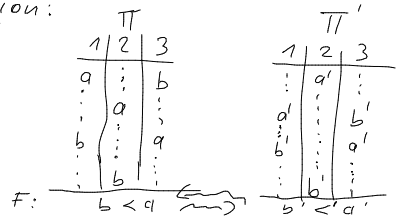
Arrow's Theorem

Lemma (Pairwise neutrality)

Assume F satisfies unanim. and IIA.
Let $\Pi = (<_1, \dots, <_n)$ and $\Pi' = (<'_1, \dots, <'_n)$ be preference profiles with $a <_i b$ iff $a' <'_i b'$ for all $i = 1, \dots, n$. Let $< = F(\Pi)$ and $<' = F(\Pi')$.
Then $a < b$ iff $a' <' b'$.

Remark: Very similar to IIA, but more general.
Visualization:

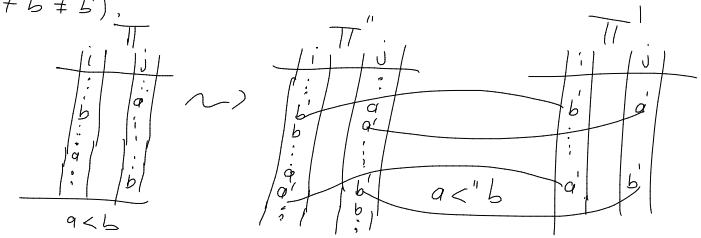


Proof:

By cases:
1) $a \neq b'$ and $a' \neq b$
2) $a = b'$ and $a' = b$
3) $a = b'$ and $a' \neq b$
4) $a \neq b'$ and $a' = b$

Case 1: Provided $a \neq b'$ and $a' \neq b$, ($a <_i b$ iff $a' <'_i b'$) implies ($a < b$ iff $a' <' b'$)

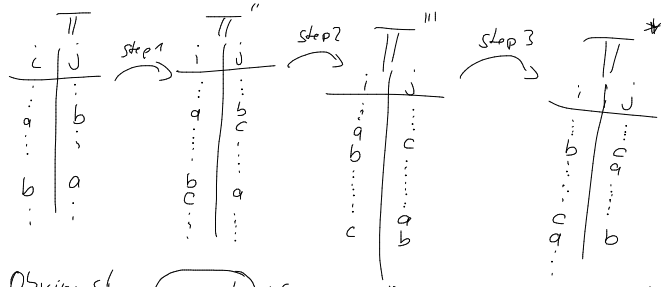
Assume that $a < b$. Construct Π'' starting from Π by moving a' just below a (if $a \neq a'$) and b' just above b (if $b \neq b'$).



Let $<'' = F(\Pi'')$. By IIA and our assumption $a < b$: $a <'' b$. Unanimity implies $a' \leq'' a$ and $b \leq'' b'$. By transitivity we get $a' <'' b'$. By IIA, we get $a' <' b'$. By symmetry, we can show that $a' <' b'$ implies $a < b$. So we get in total: $a < b$ iff $a' <' b'$.

Case 2: ($a <_i b$ iff $a' <'_i b'$) implies ($a < b$ iff $a' <' b'$) under the assumption that $a = b'$ and $a' = b$.

→ That is ($a <_i b$ iff $b <'_i a$) implies ($a < b$ iff $b <' a$).
Let c be a third alternative. Construct Π'' from Π by putting c directly below b. Construct Π''' from Π'' by moving b directly below a. Constructed Π^* from Π''' by moving a directly below c.



Obviously, ($a <_i b$) iff $b <^*_i a$. By assumption, we have $b <_i a$ iff $b <^*_i a$.

Note: step 1 implies (by case 1): $a < b$ iff $a <'' c$
step 2 implies " : $a <'' c$ iff $b <''' c$
step 3 " " : $b <''' c$ iff $b <^* a$

From that it follows $a < b$ iff $b <' a$.

Case 3: homework

Theorem (Arrow)

Every social welfare function over more than 2 alternatives that satisfies <u>unanimity</u> and <u>IIA</u> is dictatorial.

Proof: Consider two alternatives $a, b \in A$, $a \neq b$. Let us construct a sequence of profiles Π^i $(i = 0, \dots, n)$ of preference orders such that in Π^i exactly the first i voters prefer b over a.

	Π^0		Π^{i^*-1}	Π^{i^*}	...	Π^n
1:	$b <_1 a$		$a <_1 b$	$a <_1 b$		$a <_1 b$
$\vdots$						
i^*-1	$b <_{i^*-1} a$		$a <_{i^*-1} b$	$a <_{i^*-1} b$		$a <_{i^*-1} b$
i^*	$b <_{i^*} a$		$b <_{i^*} a$	$a <_{i^*} b$		$a <_{i^*} b$
i^*+1	$b <_{i^*+1} a$		$b <_{i^*+1} a$	$b <_{i^*+1} a$		$a <_{i^*+1} b$
$\vdots$						
n	$b <_n a$		$b <_n a$	$b <_n a$		$a <_n b$
	$b < a$	.....	$b < a$	$a < b$	.....	$a < b$

There must be an index i^* that is the first index such $b < a$ in $F(\Pi^{i^*-1})$ and $a < b$ in $F(\Pi^{i^*})$.

What we will show: i^* is a dictator!

Consider two alternatives $c, d \in A$ and show that for all $<_1, \dots, <_n \in L$, $\boxed{c <_{i^*} d}$ implies $c < d$, where $< = F(<_1, \dots, <_{i^*}, \dots, <_n)$.

Consider $e \notin \{c, d\}$ and construct a new preference profile $\Pi' = (<'_1, \dots, <'_n)$:

Depending on whether

	$c <_j d$	$d <_j c$
for $j < i^*$	$e <'_j c <'_j d$	$e <'_j d <'_j c$
for $j = i^*$	$c <'_j e <'_j d$	~~$d <'_j e <'_j c$~~
for $j > i^*$	$c <'_j d <'_j e$	$d <'_j c <'_j e$

Let $<' = F(\Pi')$. By IIA, we have that $c < d$ iff $c <' d$.

Let us have a look what preference order we have between (e, c) and (e, d)

	Π'	Π^{i^*-1}	Π'	Π^{i^*}
1:	$e <'_1 c$	$a <_1 b$	$e <'_1 d$	$a <_1 b$
$\vdots$			$\vdots$	$\vdots$
i^*-1	$e <'_{i^*-1} c$	$a <_{i^*-1} b$	$e <_{i^*-1} d$	$a <_{i^*-1} b$
i^*	$c <'_{i^*} e$	$b <_{i^*} a$	$e <_{i^*} d$	$a <_{i^*} b$
				$b <_{i^*+1} a$
$\vdots$			$\vdots$	$\vdots$
n:	$c <'_n e$	$b <_n a$	$d <_n e$	$b <_n a$
F()	$c < e$	$b < a$	$e <' d$	$a < b$

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pairwise neutrality

By transitivity, we get $c <' d$. With IIA we get $c < d$. And this means $c <_{i^*} d$ implies $c < d$. That is i^* is a dictator. $\square$