

Axioms for Voting Systems

We discuss them for Plurality, IRV, Borda, Schulze.

Neutrality: "Names" of candidates should not be relevant.

Anonymity: "Names" of the voters should not be relevant.

N+R: hold in PV, IRV, Borda, Schulze

Borda, Plurality : -

↳ follows from majority.

A: 4, B, C, D, E, F, G: 1
each

S = {B, C, D, E, F, G}

IRV, Schulze : +

Majority criterion: A candidate who is ranked highest by the majority (> 50%) of voters, should be the winner.

Borda : -

Plurality, IRV, Schulze : +

Mutual majority criterion: If a majority of voters prefer candidates from one subset S over all other candidates, then the winner should be from S.

Independence of Irrelevant Alternatives (IIA):

The outcome never changes if a non-winning candidate is added or removed.

Borda : -

Plurality : - $\begin{bmatrix} A: 7 \\ B: 6 \\ C: \text{stays 2 of D's votes} \end{bmatrix}$

IRV : -

Schulze : -

Independence of Clone Alternatives

A set of clones is a subset of candidates S such that all other candidates are ranked higher or lower than all candidates from S by all voters.

Schulze, IRV : +

Borda, Plurality : -

↑
finer
except

L₁: 30% } clones
L₂: 30% }
R : 40%

Monotonicity criterion: If a candidate wins in an election, then he will still win, if one voter ranks him higher.

Borda, Plurality, Schulze : +

IRV : -

Consistency criterion: If there are two sets of voters, with separate elections that have the same winner, then the combined election should have the same winner.

Borda, Plurality : +

Schulze, IRV : -

Polynomial-time computability: The winner should be computable in time polynomial in the number of candidates and linear in the number of voters.

Borda, Plurality, IRV, Schulze : +

Dodgson : -

Resolvability:

- 1) For every tie between "winners", one vote should resolve the tie.
- 2) The proportion of preference profiles leading to a tie should approach zero when the number of voters approaches infinity.

Plurality, IRV, Borda, Schulze : +

Theorem (May, 1958): A voting method for two alternatives satisfies anonymity, neutrality, and monotonicity, if it is the plurality method.

Prof: " \leq " : obvious.

" \Rightarrow " : For simplicity, we assume that the number of voters is odd.

(Case 1) Candidate a wins.

Then by monotonicity, a still wins whenever $|A| > |B|$. That's the plurality method. (Because, with neutrality, we also get b as a winner if $|B| > |A|$).

(Case 2) Candidate b wins. Assume that one voter for a changes his preference to b. Then $|A'| + 1 = |B'|$. By monotonicity, b must still win. This is completely symmetric to original vote, hence, by neutrality, a should win. \square

Anonymity + neutrality \Rightarrow only the numbers of voters for a candidate count.

Let A be the set of voters that prefer candidate a, and let B be the set of voters that prefer candidate b. Consider a vote with $|A| = |B| + 1$.

For three or more alternatives, there are no voting methods that satisfy a small set of reasonable criteria.

Arrow's Impossibility Theorem

Theorem (Arrow): Every social welfare function over more than two alternatives that satisfies neutrality and IID is necessarily a dictatorship.

Def.: (Total maximity)

For all $\prec \in L$: $F(\prec, \prec, \dots, \prec) = \prec$.

Def.: (Partial maximity)

For all $\prec_1, \prec_2, \dots, \prec_n, \prec \in L$ with
 $\prec = F(\prec_1, \dots, \prec_n)$, if $a \prec_i b$ for
all $i = 1, \dots, n$, then also $a \prec b$.

Remark. Partial maximity implies total
maximity, but not the other way around.