

Background: Linear Programming

Goal: Solve a system of linear inequalities over n real-valued variables while maximizing/minimizing some linear objective function.

Example:

	Cutting	Assembly	Postproc.	Profit/item
x sort 1	25	60	68	30 €
y sort 2	75	60	34	40 €
constraint (day)	≤ 450	≤ 480	≤ 476	\uparrow MAXIMIZE

290 €

Goal: Find numbers of pieces/items of sorts 1 (x) and 2 (y) produced per day such that the resource constraints are met and the profit is maximized.

Formulation: $x \geq 0, y \geq 0$ (1)

$$25 \cdot x + 75 \cdot y \leq 450 \quad (2)$$

$$60 \cdot x + 60 \cdot y \leq 480 \quad (3)$$

$$68 \cdot x + 34 \cdot y \leq 476 \quad (4)$$

$$\text{maximize } 30 \cdot x + 40 \cdot y \quad (5)$$

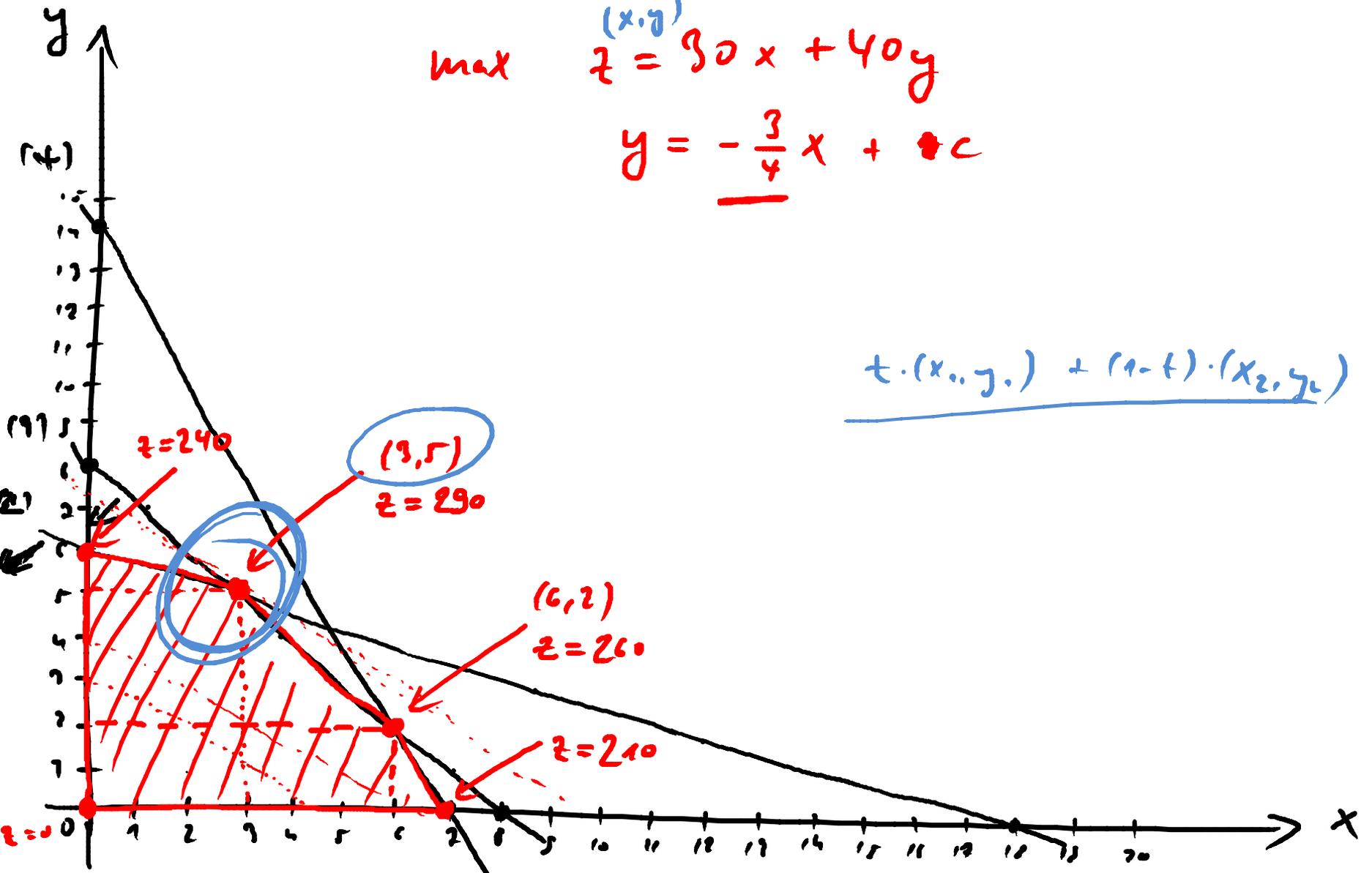
(1) - (4) : feasible solutions

(5) : objective function.

$$\max_{(x,y)} z = 30x + 40y$$

$$y = -\frac{3}{4}x + c$$

$$t \cdot (x_1, y_1) + (1-t) \cdot (x_2, y_2)$$



(2) $x + 3y \leq 18$
 $\Leftrightarrow y \leq 6 - \frac{1}{3}x$

(1) $x + y \leq 8$
 $\Leftrightarrow y \leq 8 - x$

(4) $2x + y \leq 14$
 $\Leftrightarrow y \leq 14 - 2x$

Def: A linear program (LP) in standard form consists of

- n real-valued variables x_i

- n coefficients b_i

- m constants c_j

- $n \cdot m$ coefficients a_{ji}

- m inequalities $c_j \leq \sum_{i=1}^n a_{ji} x_i$

(for $j=1, \dots, m$)

- an objective function $\sum_{i=1}^n b_i x_i$

to be minimized.

Remark: • Maximization instead of minimization:

change the signs of all b_i 's.

• Equalities: $x + y \leq c$ if ex. $z \geq 0$

$$\text{s.t. } x + y + z = c.$$

(z is called a slack variable).

LP solving algorithms: Usually, one uses the

Simplex algorithm (worst-case exponential);

simplex algorithm is still often preferred in practice

over existing polynomial algorithms.

→ lp-solve

Encoding of finite 2SG MSWE as LP

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where

- $N = \{1, 2\}$
- A_1, A_2 are finite
- $u_1(\alpha, \beta) = -u_2(\alpha, \beta)$ for $\alpha \in \Delta(A_1), \beta \in \Delta(A_2)$.

Maximinax Theorem: $NE \Rightarrow$ pair MM.

pair of MM

mixed strategies

some NE ex.
 \implies NE

Nash's th.

\implies

some NE ex.

Mixed strat.

\implies

NE \Leftrightarrow pairs of MM

Hence, to find a MSNE, look for pairs of
(mixed-strategy) MM.

Assume that player 1 seeks a MM α_1 .

For each $\alpha_1 \in \Delta(A_1)$ of player 1:

• determine utility under player 2's

best response

sufficient to consider pure responses.

Maximize over these utilities.

LP constraints:

$$\alpha_n(a_n) \geq 0$$

for all $a_n \in A_1$

$$\sum_{a_n \in A_1} \alpha_n(a_n) = 1$$

variables of LP



$$\underbrace{U_n(\alpha_n, b)} \geq u$$

for all $b \in A_2$.

$$\sum_{a_n \in A_1} \alpha_n(a_n) \cdot u_n(a_n, b)$$

constant
in \mathbb{R} .

Maximize u .

- A solution to this LP is a MM for player 1.
 - A solution to a similar LP for player 2 is a MM for player 2.
-

Example: Matching pennies

		pl. 2	
		H	T
pl. 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

LP for player 1:

$$\alpha_1(H) \geq 0, \quad \alpha_1(T) \geq 0$$

$$\alpha_1(H) + \alpha_1(T) = 1$$

$$1 \cdot \alpha_1(H) - 1 \cdot \alpha_1(T) \geq u$$

$$-1 \cdot \alpha_1(H) + 1 \cdot \alpha_1(T) \geq u$$

Maximize u subject to these four constraints.

Soln:

$$\alpha_1(H) = \alpha_1(T) = 1/2$$

Remark: Alternative (but slower) encoding
using minimaximization instead of maxi-
minimization possible. LP with inequalities:

$$U_1(a_1, \beta) \leq u \quad \text{for each } a_1 \in A_1.$$

and "minimize u " as objective function.

Next step: Do the same thing for

non-two-sum games.

Instead of an LP, we use Linear Complementarity Problems (LCP):

- In LCP, there is no objective function.

- In LCP, one has so-called complementarity constraints; for two vectors of variables,

(x_1, \dots, x_n) and (y_1, \dots, y_n) , there are

constraints $x_i \cdot y_i = 0$ (f.o. $i=1, \dots, n$).

Let $G = \langle \{1, 2\}, (A_1, A_2), (u_1, u_2) \rangle$ be a general finite strategic game with $A_1 = \{a_1, \dots, a_m\}$ and $A_2 = \{b_1, \dots, b_n\}$. Suppose that (α, β) is a MNE of G with payoff profile (u, v) . Then the constraints are:

$$\left. \begin{array}{l} \alpha(a_i) \geq 0 \quad (i=1, \dots, m) ; \quad \sum_{i=1}^m \alpha(a_i) = 1 \\ \beta(b_j) \geq 0 \quad (j=1, \dots, n) ; \quad \sum_{j=1}^n \beta(b_j) = 1 \end{array} \right\} \textcircled{**}$$

$$u - u_1(a_i, \beta) \geq 0 \quad (i=1, \dots, m)$$

$$v - u_2(\alpha, b_j) \geq 0 \quad (j=1, \dots, n)$$

$$\alpha(a_i) \cdot (u - u_1(a_i, \beta)) = 0 \quad (i=1, \dots, m)$$

$$\beta(b_j) \cdot (v - u_2(\alpha, b_j)) = 0 \quad (j=1, \dots, n)$$

$\neq 0$
 \neq
 $a_i \notin \text{supp}(\alpha)$

$= 0 \iff a_i \in B_1(\beta)$

$\textcircled{**}$

Proposition: A mixed strategy profile (α, β) with payoff profile (u, v) is a MSNE in G iff there exists a solution to the above LCP with variables $u, v, \alpha(a_1), \dots, \alpha(a_m), \beta(b_1), \dots, \beta(b_n)$.

Proof: " \Rightarrow ": Let (α, β) be a MSNE with payoffs (u, v) . By the support lemma, for each player and for each strategy in the support of his mixed strategy, this is a best response to the other player's mixed strategy. Therefore, constraints $\textcircled{*}$ are satisfied.

Constraints $(**)$ (prob. distr.) are trivially satisfied by MSNE strategies.

" \Leftarrow ": Assume, we have a solution to the LCP. Because of $(**)$, α and β must be mixed strategies. For all $a_i \in A_1$, (either) $a_i \notin \text{supp}(\alpha)$, or $a_i \in B_1(\beta)$.

In addition, u is the best utility player 1 can get against β using a pure strategy.

Hence, u is the utility player 1 gets for his best response against β . Similar argument for player 2. $\Rightarrow (\alpha, \beta)$ MSNE with payoffs (u, v) . \square

Naive approach to solving LCPs:

① Enumerate all pairs of possible supports:

$$(2^m - 1) \cdot (2^n - 1) \text{ such pairs.}$$

② For each such pair $(\text{supp}(\alpha), \text{supp}(\beta))$, simplify / convert to LP the LCP as follows:

Replace conditions of the form

$$\alpha(a_i) \cdot (u - U_n(a_i, \beta)) = 0 \text{ by}$$

$$\begin{cases} u - U_n(a_i, \beta) = 0 & \text{if } a_i \in \text{supp}(\alpha) \\ \alpha(a_i) = 0 & \text{if } a_i \notin \text{supp}(\alpha). \end{cases}$$

Same thing for player 2. Trivial objective function: maximize 0.

Then, we have a LP. We can solve this using any LP solver and use solutions to LPs as solutions to the original LP.

Lenke-Houson algorithm is a direct way of solving such games.