Background: Linear Programming

Goal: Solve a system of linear inequalities over n real-valued variables while maximizing/minimizing some linear objective function.

Example:

		Cutting	Assembly	Packaging	Profit/item
3 x	sort 1	25	60	68	30 €
5 y	sort 2	75	60	34	40 €
	constraint (day)	≤ 450	≤ 480	≤ 476	↑ MAXIMIZE

290 €

Goal: Find numbers of pieces/items of sorts 1 (x) and 2 (y) produced per day such that the resource constraints are met and the profit is maximized.

Formulation:

$$x \geq 0, \quad y \geq 0 \qquad (1)$$

$$25 \cdot x + 75 \cdot y \leq 450 \qquad (2)$$

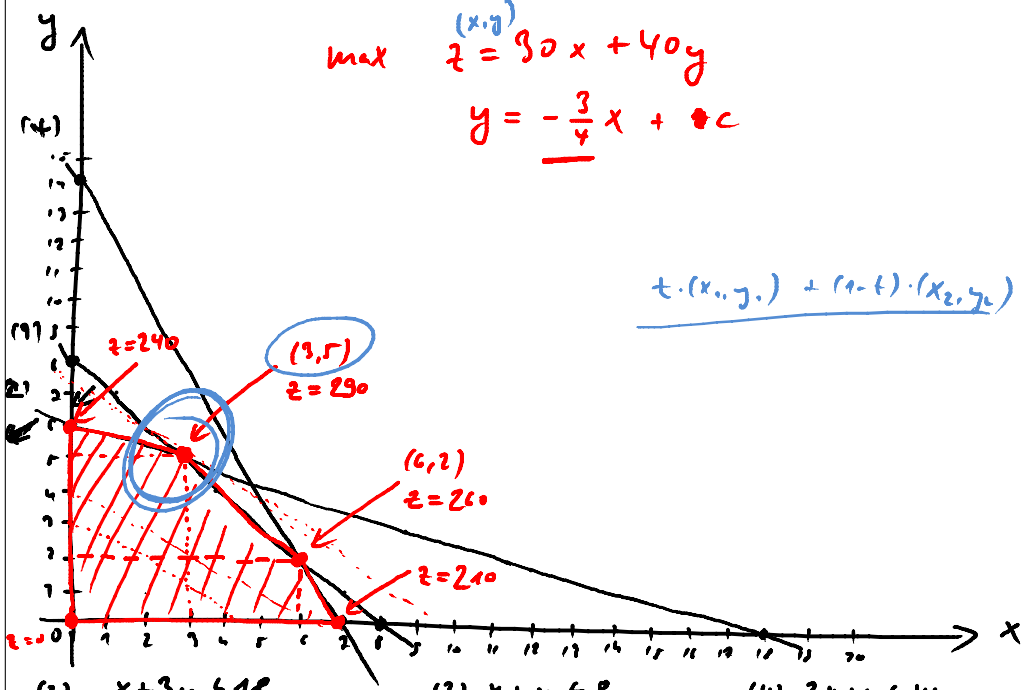
$$60 \cdot x + 60 \cdot y \leq 480 \qquad (3)$$

$$68 \cdot x + 34 \cdot y \leq 476 \qquad (4)$$

$$\text{maximize } 30 \cdot x + 40 \cdot y \qquad (5)$$

(1)–(4): feasible solutions

(5): objective function.



(2) $x + 3y \leq 18$ $\Leftrightarrow y \leq 6 - \frac{1}{3}x$

(3) $x + y \leq 8$ $\Leftrightarrow y \leq 8 - x$

(4) $2x + y \leq 14$ $\Leftrightarrow y \leq 14 - 2x$

Def: A linear program (LP) in standard form consists of

- n real-valued variables x_i
- n coefficients b_i
- m constants c_j
- $n \cdot m$ coefficients a_{ji}
- m inequalities $c_j \leq \sum_{i=1}^{n} a_{ji} x_i$ (for $j = 1, \ldots, m$)
- an objective function $\sum_{i=1}^{n} b_i x_i$ to be minimized.

Remark: • Maximization instead of minimization: change the signs of all b_i's.

• Equalities: $x + y \le c$ iff ex. $z \ge 0$ s.t. $x + y + z = c$.

(z is called a slack variable).

LP solving algorithms: Usually, one uses the simplex algorithm (worst-case exponential); simplex algorithm is still often preferred in practice over existing polynomial algorithms.

$\rightsquigarrow$ $\boxed{\text{lp\_solve}}$

Encoding of finite 2SG MSNE as LP

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where

- $N = \{1, 2\}$
- A_1, A_2 are finite
- $u_1(\alpha, \beta) = -u_2(\alpha, \beta)$ for $\alpha \in \Delta(A_1)$, $\beta \in \Delta(A_2)$.

Maximin-minimax theorem: NE $\Rightarrow$ pair MM.

pair of MM $\xrightarrow{\text{some NE ex.}}$ NE

mixed strategies $\xrightarrow{\text{Nash's th.}}$ some NE ex.

$\xrightarrow{\text{Mixed Nash.}}$ NE $\Leftrightarrow$ pairs of MM

Hence, to find a MSNE, look for pairs of (mixed-strategy) MM.

Assume that player 1 seeks a MM α_1.

For each $\alpha_1 \in \Delta(A_1)$ of player 1:

- determine utility under player 2's best response

sufficient to consider pure responses.

Maximize over these utilities.

LP constraints:

$\boxed{\alpha_1(a_1)} \ge 0$ for all $a_1 \in A_1$

$\sum_{a_1 \in A_1} \boxed{\alpha_1(a_1)} = 1$

variables of LP

$\Rightarrow$ $\underbrace{U_1(\alpha_1, b)}_{\sum_{a_1 \in A_1} \boxed{\alpha_1(a_1)} \cdot \boxed{u_1(a_1, b)}} \ge \boxed{u}$ for all $b \in A_2$.

constant in $\mathbb{R}$.

Maximize $\boxed{u}$.

- A solution to this LP is a MM for player 1.
- A solution to a similar LP for player 2 is a MM for player 2.

Example: Matching pennies

		pl. 2: H	pl. 2: T
pl. 1	H	1, −1	−1, 1
	T	−1, 1	1, −1

LP for player 1:

$$\alpha_1(H) \geq 0, \quad \alpha_1(T) \geq 0$$

$$\alpha_1(H) + \alpha_1(T) = 1$$

$$1 \cdot \alpha_1(H) - 1 \cdot \alpha_1(T) \geq u$$

$$-1 \cdot \alpha_1(H) + 1 \cdot \alpha_1(T) \geq u$$

Maximize u subject to these four constraints.

Sol.: $\alpha_1(H) = \alpha_1(T) = 1/2$

Remark: Alternative (but similar) encoding using minimaximization instead of maxi-minimization possible. LP with inequalities:

$$U_1(a_1, \beta) \leq u \quad \text{for each } a_1 \in A_1.$$

and „minimize u" as objective function.

Next step: Do the same thing for non-zero-sum games.

Instead of an LP, we use Linear Complementarity Problems (LCP):

- In LCP, there is no objective function.
- In LCP, one has so-called <u>complementarity constraints</u>; for two vectors of variables, $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$, there are constraints $x_i \cdot y_i = 0$ (f.a. $i = 1, \dots, n$).

Let $G = \langle \{1,2\}, (A_1, A_2), (u_1, u_2) \rangle$ be a general finite strategic game with $A_1 = \{a_1, \dots, a_m\}$ and $A_2 = \{b_1, \dots, b_n\}$. Suppose that (α, β) is a MSNE of G with payoff profile (u, v). Then these constraints hold:

$$\left.\begin{array}{ll} \alpha(a_i) \geq 0 \quad (i=1,\dots,m) \; ; & \sum_{i=1}^{m} \alpha(a_i) = 1 \\ \beta(b_j) \geq 0 \quad (j=1,\dots,n) \; ; & \sum_{j=1}^{n} \beta(b_j) = 1 \end{array}\right\} (**)$$

$$\left.\begin{array}{ll} u - U_1(a_i, \beta) \geq 0 & (i=1,\dots,m) \\ v - U_2(\alpha, b_j) \geq 0 & (j=1,\dots,n) \\ \boxed{\alpha(a_i)} \cdot \boxed{(u - U_1(a_i, \beta))} = 0 & (i=1,\dots,m) \\ \beta(b_j) \cdot (v - U_2(\alpha, b_j)) = 0 & (j=1,\dots,n) \end{array}\right\} (*)$$

$= 0$ iff $a_i \notin \mathrm{supp}(\alpha)$

$= 0$ iff $a_i \in B_1(\beta)$

Constraints $(**)$ (prob. distr.) are trivially satisfied by MSNE strategies.

"$\Leftarrow$": Assume, we have a solution to the LCP. Because of $(**)$, α and β must be mixed strategies. For all $a_i \in A_1$, (either) $a_i \notin \mathrm{supp}(\alpha)$, or $a_i \in B_1(\beta)$. In addition, u is the best utility player 1 can get against β using a pure strategy. Hence, u is the utility player 1 gets for his best responses against β. Similar argument for player 2. $\Rightarrow (\alpha, \beta)$ MSNE with payoffs (u, v). $\square$

Proposition: A mixed strategy profile (α, β) with payoff profile (u, v) is a MSNE in G iff there exists a solution to the above LCP with variables $u, v, \alpha(a_1), \dots, \alpha(a_m), \beta(b_1), \dots, \beta(b_n)$.

Proof: "$\Rightarrow$": Let (α, β) be a MSNE with payoffs (u, v). By the support lemma, for each player and for each pure strategy in the support of his mixed strategy, this is a best response to the other player's mixed strategy. Therefore, constraints $(*)$ are satisfied.

Naive approach to solving LCPs:

1. Enumerate all pairs of possible supports: $(2^m - 1) \cdot (2^n - 1)$ such pairs.
2. For each such pair $(\mathrm{supp}(\alpha), \mathrm{supp}(\beta))$, simplify / convert to LP the LCP as follows: Replace conditions of the form $\alpha(a_i) \cdot (u - U_1(a_i, \beta)) = 0$ by

$$\begin{cases} u - U_1(a_i, \beta) = 0 & \text{if } a_i \in \mathrm{supp}(\alpha) \\ \alpha(a_i) = 0 & \text{if } a_i \notin \mathrm{supp}(\alpha). \end{cases}$$

Same thing for player 2. Trivial objective function: maximize 0.

Then, we have a LP. We can solve this using any LP solver and use solutions to LPs as solutions to the original LCP.

Lemke-Howson algorithm is a direct way of solving such games.