Existence of MSNE

Theorem (Nash): Every finite strategic game has a MSNE.

Proof: later

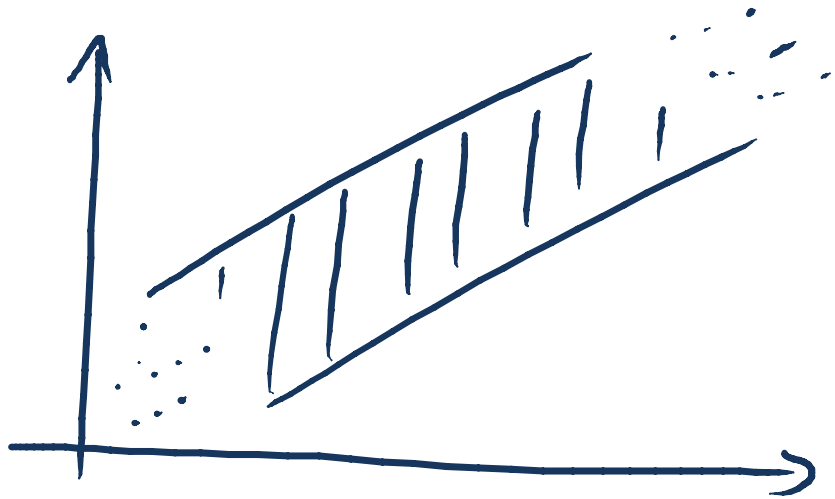
Preliminaries.

Def.: (a) $X \subseteq \mathbb{R}^n$ is <u>bounded</u> if for $1 \leq i \leq n$ ex. $a_i, b_i \in \mathbb{R}$ such that

$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

closed intervals

Counterexample:



not bounded.

(b) $X \subseteq \mathbb{R}^n$ is <u>closed</u> if the limit of each convergent sequence of elements of X is contained in X.

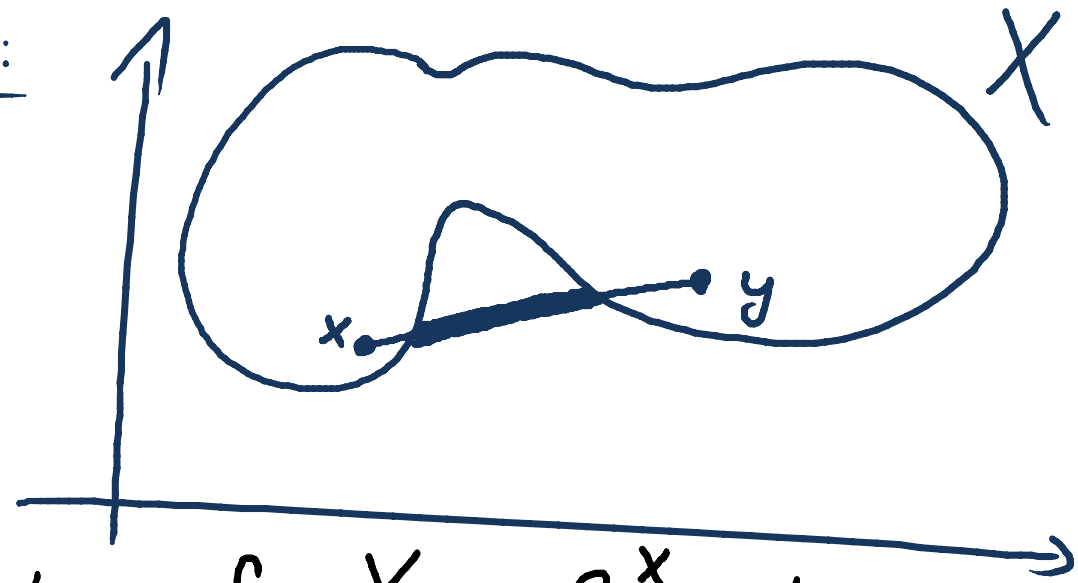
Counterex:

$[0,1)$ not closed

Sequence: $1-\frac{1}{2},\ 1-\frac{1}{3},\ 1-\frac{1}{4},\ \dots \xrightarrow{n\to\infty} 1 \notin [0,1)$

(c) $X \subseteq \mathbb{R}^n$ is <u>convex</u> if for each $x, y \in X$, and for any $t \in [0,1]$, also $(1-t)\cdot x + t\cdot y \in X$.

<u>Counterex.:</u>



(d) For function $f: X \to 2^X$, the <u>graph</u> of f is the set $\mathrm{Graph}(f) = \{(x,y) \mid x \in X,\ y \in f(x)\}$.

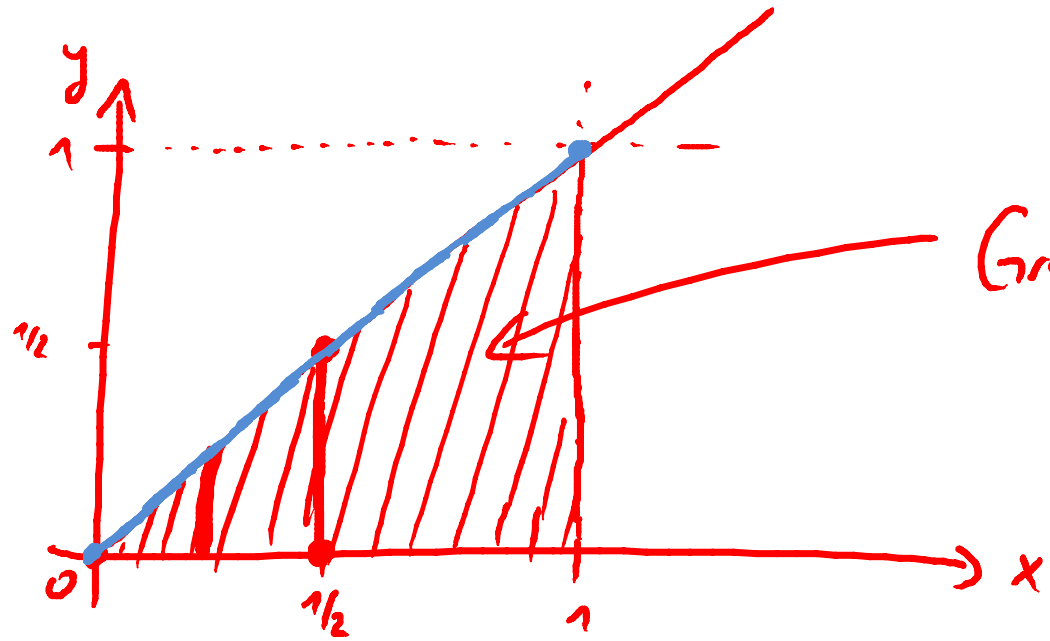
<u>Theorem (Kakutani)</u>: Let $X \subseteq \mathbb{R}^n$ be nonempty, closed, bounded, and convex set and $f : X \to 2^X$ be a function such that:

(i) for each $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and

(ii) $\mathrm{Graph}(f)$ is closed.

Then ex. $x \in X$ with $x \in f(x)$, i.e. f has a fixpoint. $\square$

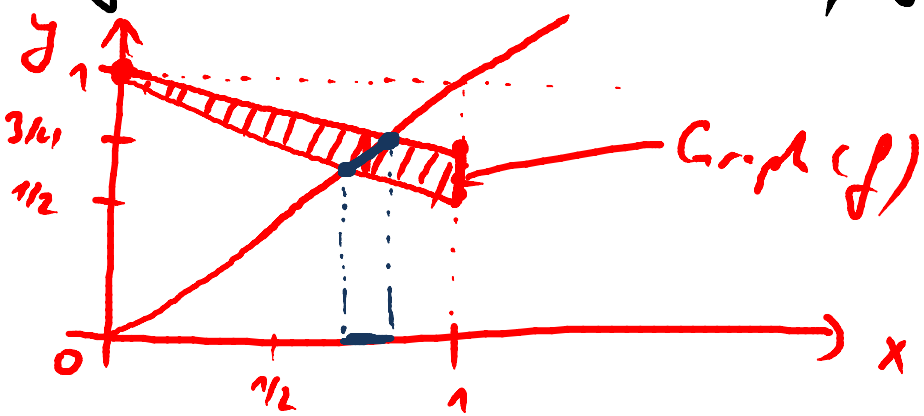
Example(s): (a) $f : [0,1] \to 2^{[0,1]}$, $f(x) = \{y \mid y \leq x\}$.



$x \in f(x)$

(e.g. $1/2 \in \{y \mid y \leq 1/2\}$)

(b) $f : [0,1] \to 2^{[0,1]}$, $f(x) = \{y \mid 1 - \frac{x}{2} \leq y \leq 1 - \frac{x}{4}\}$.



$f(0) = [1, 1]$

$f(1) = [1 - 1/2,\ 1 - 1/4]$