

(b) With (I) : $u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y)$.

$$u_1(x^*, y^*) = -u_2(x^*, y^*) = -\max_{y \in A_2} \min_{x \in A_1} u_2(x, y)$$

$$= \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$$

(c) Let x^*, y^* be MM of players 1 and 2,

$$\text{and } \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$$

$$=: v^*$$

$$\Rightarrow -v^* = \max_{y \in A_2} \min_{x \in A_1} u_2(x, y)$$

$$x^*, y^* \text{ MM} \Rightarrow u_1(x^*, y) \geq v^* \quad \text{f.o. } y \in A_2$$

and

$$u_2(x, y^*) \geq -v^* \quad \text{f.o. } x \in A_1.$$

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With $y = y^*$ and $x = x^*$:

$$u_1(x^*, y^*) \geq v^*$$

$$u_2(x^*, y^*) \geq -v^*$$

$$\stackrel{u_1 = -u_2}{\Rightarrow} u_1(x^*, y^*) \leq v^*$$

$$\Rightarrow u_1(x^*, y^*) = v^*$$

$$\begin{aligned}
 (*) \wedge (***) &\Rightarrow u_1(x^*, y) \geq u_1(x^*, y^*) \text{ f.a. } y \in R_1 \\
 &\Rightarrow u_2(x^*, y) \leq u_2(x^*, y^*) \text{ f.a. } y \in R_2 \\
 &\Rightarrow y^* \in B_2(x^*) .
 \end{aligned}$$

$$(***) \wedge (***) \Rightarrow \dots \Rightarrow x^* \in B_1(y^*) .$$

$$\Rightarrow (x^*, y^*) \text{ is a NE } \quad \square$$

Mixed strategies

Motivating example: Matching pennies

		pl. 2	
		H ^{1/3}	T ^{2/3}
pl. 1	H ^{2/3}	1, -1	-1, 1
	T ^{1/3}	-1, 1	1, -1

For player 1: $\alpha_1(H) = \frac{2}{3}$, $\alpha_1(T) = \frac{1}{3}$

For player 2: $\alpha_2(H) = \frac{1}{3}$, $\alpha_2(T) = \frac{2}{3}$

Def: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game. Let $\Delta(A_i)$ be the set of probability distributions over A_i .

Then an $\alpha_i \in \Delta(A_i)$ is a mixed strategy of player i in G , where $\alpha_i(a_i)$ is the probability that player i chooses $a_i \in A_i$.

A profile $(\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ induces a probability distribution over $A = \prod_{i \in N} A_i$

as $p(a) = \prod_{i \in N} \alpha_i(a_i)$.

Then (in example above):

$$p(H, H) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}, \quad u_1(H, H) = +1$$

$$p(H, T) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}, \quad u_1(H, T) = -1$$

$$p(T, H) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}, \quad u_1(T, H) = -1$$

$$p(T, T) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}, \quad u_1(T, T) = +1$$

$$\begin{aligned} u_1, \text{ expected} &: \frac{2}{9} \cdot (+1) + \frac{4}{9} \cdot (-1) + \frac{1}{9} \cdot (-1) + \frac{2}{9} \cdot (+1) \\ &= -\frac{1}{9} \end{aligned}$$

$$u_2, \text{ expected} : +\frac{1}{9}$$

Def.: Let $\alpha \in \prod_{i \in N} \Delta(A_i)$. Then the expected utility of α for player i is

$$U_i(\alpha) = \sum_{a \in A} \underbrace{p(a)}_{\prod_i \alpha_i(a_i)} \cdot u_i(a)$$

Ex. 1 $U_1(\alpha_1, \alpha_2) = -\frac{1}{9}$

$$U_2(\alpha_1, \alpha_2) = +\frac{1}{9}$$

Def.: The mixed extension of a (finite)

strategic game $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$

is the strategic game $(N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N})$

where the $\Delta(A_i)$ and U_i are defined as above.

Def.: Let α_i be a mixed strategy. The

support of α_i is the set of pure

strategies

$$\text{supp}(\alpha_i) = \{a_i \in A_i \mid \alpha_i(a_i) > 0\}.$$

Def.: Let G be a strategic game.

A Nash-equilibrium in mixed strategies

(mixed-strategy NE, MSNE) of G is

a NE of the mixed extension of G .

Support-Lemma: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$

be a finite strategic game. Then $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$

$\in \prod_{i \in N} \Delta(A_i)$ is a MSNE of G iff

for each player $i \in N$, every pure strategy in $\text{supp}(\alpha_i^*)$ is a best response to α_{-i}^* .

Example: Matching Pennies (as above)

$$\alpha_1(H) = \frac{2}{3}, \quad \alpha_1(T) = \frac{1}{3}$$

$$\alpha_2(H) = \frac{1}{3}, \quad \alpha_2(T) = \frac{2}{3}$$

MSNE?

$$\begin{aligned} U_2(\alpha_1, H) &= \alpha_1(H) \cdot u_2(H, H) \\ &\quad + \alpha_1(T) \cdot u_2(T, H) \\ &= \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot (+1) = -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} U_2(\alpha_1, T) &= \alpha_1(H) \cdot u_2(H, T) \\ &\quad + \alpha_1(T) \cdot u_2(T, T) \\ &= \frac{2}{3} \cdot (+1) + \frac{1}{3} \cdot (-1) = +\frac{1}{3} \end{aligned}$$

$H \in \text{supp}(\alpha_2)$, but $H \notin \mathcal{B}_2(\alpha_1)$
 $\stackrel{\text{supp-lemma}}{\implies} (\alpha_1, \alpha_2) \underline{\text{not}} \in \text{MSNE}.$

Proof (support lemma): Let α^* be a MSNE

and $a_i \in \text{supp}(\alpha_i^*)$.

" \Rightarrow ": Suppose that a_i is not a best response to α_{-i}^* . Then $\exists a_i' \in A_i$ s.t.

$$U_i(\alpha_{-i}^*, a_i') > U_i(\alpha_{-i}^*, a_i).$$

Then a_i' with weight shifted from a_i to a_i' would be a better response to α_{-i}^* than a_i^* is. Then $\alpha_i^* \notin B_i(\alpha_{-i}^*)$. Then α^* is not a MSNE. \downarrow

" \Leftarrow ": Suppose that α^* is not a MSNE.

Then there exists a player $i \in N$ and a mixed strategy α'_i such that

$$U_i(\alpha_{-i}^*, \alpha'_i) > U_i(\alpha_{-i}^*, \alpha_i^*).$$

Then (because of the linearity of U_i) there exists a strategy $a'_i \in A_i$ in $\text{supp}(\alpha'_i)$ with higher payoff against α_{-i}^* than at least one pure strategy $a_i \in \text{supp}(\alpha_i^*)$.

I.e. not all $a_i \in \text{supp}(\alpha_i^*)$ are best responses to α_{-i}^* . \Downarrow □