(b) With (I): $u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x,y)$.

$$u_1(x^*, y^*) = -u_2(x^*, y^*) = -\max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$$

$$= \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$$

(c) Let x^*, y^* be MM of players 1 and 2,

and $\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$

$$=: v^*$$

$$\Rightarrow -v^* = \max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$$

x^*, y^* MM $\Rightarrow u_1(x^*, y) \geq v^*$ f.a. $y \in A_2$ (*)

and $u_2(x, y^*) \geq -v^*$ f.a. $x \in A_1$. (**)

With $y = y^*$ and $x = x^*$:

$$u_1(x^*, y^*) \geq v^*$$

$$\underbrace{u_2(x^*, y^*) \geq -v^*}$$

$$\overset{u_1 = -u_2}{\Rightarrow} u_1(x^*, y^*) \leq v^*$$

$$\Rightarrow u_1(x^*, y^*) = v^* \quad (***)$$

(*) $\wedge$ (***) $\Rightarrow u_1(x^*, y) \geq u_1(x^*, y^*)$ f.a. $y \in A_2$

$\Rightarrow u_2(x^*, y) \leq u_2(x^*, y^*)$ f.a. $y \in A_2$

$\Rightarrow y^* \in B_2(x^*)$.

(**) $\wedge$ (***) $\Rightarrow \dots \Rightarrow x^* \in B_1(y^*)$.

$\Rightarrow (x^*, y^*)$ is a NE $\square$

Mixed strategies

Motivating example: Matching pennies

		pl. 2: H (1/3)	pl. 2: T (2/3)
pl. 1	H (2/3)	1, −1	−1, 1
	T (1/3)	−1, 1	1, −1

For player 1: $\alpha_1(H) = \frac{2}{3}$, $\alpha_1(T) = \frac{1}{3}$

For player 2: $\alpha_2(H) = \frac{1}{3}$, $\alpha_2(T) = \frac{2}{3}$

Def.: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game. Let $\Delta(A_i)$ be the set of probability distributions over A_i.
Then an $\alpha_i \in \Delta(A_i)$ is a <u>mixed strategy</u> of player i in G, where $\alpha_i(a_i)$ is the probability that player i chooses $a_i \in A_i$.
A profile $(\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ induces a probability distribution over $A = \prod_{i \in N} A_i$ as $p(a) = \prod_{i \in N} \alpha_i(a_i)$.

Then (in example above):

$$p(H,H) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}, \quad u_1(H,H) = +1$$

$$p(H,T) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}, \quad u_1(H,T) = -1$$

$$p(T,H) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}, \quad u_1(T,H) = -1$$

$$p(T,T) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}, \quad u_1(T,T) = +1$$

u_1, expected: $\frac{2}{9} \cdot (+1) + \frac{4}{9} \cdot (-1) + \frac{1}{9} \cdot (-1) + \frac{2}{9} \cdot (+1)$
$= -\frac{1}{9}$

u_2, expected: $+\frac{1}{9}$

Def.: Let $\alpha \in \prod_{i \in N} \Delta(A_i)$. Then the <u>expected utility</u> of α for player i is

$$\mathcal{U}_i(\alpha) = \sum_{a \in A} \underbrace{p(a)}_{\prod_i \alpha_i(a_i)} \cdot u_i(a)$$

Ex.: $\mathcal{U}_1(\alpha_1, \alpha_2) = -\frac{1}{9}$

$\mathcal{U}_2(\alpha, \alpha_2) = +\frac{1}{9}$

Def.: The <u>mixed extension</u> of a (finite) strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is the strategic game $\langle N, (\Delta(A_i))_{i \in N}, (\mathcal{U}_i)_{i \in N} \rangle$ where the $\Delta(A_i)$ and $\mathcal{U}_i$ are defined as above.

Def.: Let α_i be a mixed strategy. The <u>support</u> of α_i is the set of pure strategies

$$\mathrm{supp}(\alpha_i) = \{ a_i \in A_i \mid \alpha_i(a_i) > 0 \}.$$

Def.: Let G be a strategic game.
A <u>Nash-equilibrium in mixed strategies</u> (mixed-strategy NE, <u>MSNE</u>) of G is a NE of the mixed extension of G.

Support-Lemma: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game. Then $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*) \in \prod_{i \in N} \Delta(A_i)$ is a MSNE of G iff for each player $i \in N$, every pure strategy in $\mathrm{supp}(\alpha_i^*)$ is a best response to α_{-i}^*.

Proof (Support Lemma): Let α^* be a MSNE and $a_i \in \mathrm{supp}(\alpha_i^*)$.

„$\Rightarrow$": Suppose that a_i is <u>not</u> a best response to α_{-i}^*. Then $\exists\, a_i' \in A_i$ s.t.

$$\mathcal{U}_i(\alpha_{-i}^*, a_i') > \mathcal{U}_i(\alpha_{-i}^*, a_i).$$

Then α_i' with weight shifted from a_i to a_i' would be a better response to α_{-i}^* than a_i^* is. Then $\alpha_i^* \notin B_i(\alpha_{-i}^*)$. Then α^* is not a MSNE. ↯

Example: <u>Matching Pennies</u> (as above)

$$\alpha_1(H) = \tfrac{2}{3}, \quad \alpha_1(T) = \tfrac{1}{3}$$
$$\alpha_2(H) = \tfrac{1}{3}, \quad \alpha_2(T) = \tfrac{2}{3}$$

MSNE?

$$\begin{aligned}\mathcal{U}_2(\alpha_1, H) &= \alpha_1(H) \cdot u_2(H,H) + \alpha_1(T) \cdot u_2(T,H) \\ &= 2/3 \cdot (-1) + 1/3 \cdot (+1) = -1/3\end{aligned}$$

$$\begin{aligned}\mathcal{U}_2(\alpha_1, T) &= \alpha_1(H) \cdot u_2(H,T) + \alpha_1(T) \cdot u_2(T,T) \\ &= 2/3 \cdot (+1) + 1/3 \cdot (-1) = +1/3\end{aligned}$$

$H \in \mathrm{supp}(\alpha_2)$, but $H \notin B_2(\alpha_1)$
$\overset{\text{Support-Lemma}}{\Longrightarrow}$ (α_1, α_2) <u>not</u> a MSNE.

„$\Leftarrow$": Suppose that α^* is <u>not</u> a MSNE. Then there exists a player $i \in N$ and a <u>mixed</u> strategy α_i' such that

$$\mathcal{U}_i(\alpha_{-i}^*, \alpha_i') > \mathcal{U}_i(\alpha_{-i}^*, \alpha_i^*).$$

Then (because of the linearity of $\mathcal{U}_i$) there exists a strategy $a_i' \in A_i$ in $\mathrm{supp}(\alpha_i')$ with higher payoff against α_{-i}^* that at least one pure strategy $a_i \in \mathrm{supp}(\alpha_i^*)$.
I.e. not all $a_i \in \mathrm{supp}(\alpha_i^*)$ are best responses to α_{-i}^*. ↯ $\square$