

		2	
		L	R
1	T	1, -1	2, -2
3	B	-2, 2	-4, 4

$\rightarrow 1$

$\rightarrow -4$

↓                  ↓  
 -1                -2              ~~1~~       $\rightarrow \max_{a_1} \min_{a_2} u_1(\dots)$

1                2                1      //  
 $\rightarrow \min_{a_2} \max_{a_1} u_2(\dots)$

Lemma: Let  $G$  be a  $2SG$ . Then

$$\max_{y \in \bar{A}_2} \min_{x \in \bar{A}_1} u_2(x, y) = -\min_{y \in \bar{A}_2} \max_{x \in \bar{A}_1} u_1(x, y).$$

Proof: For each real-valued function  $f$ , it holds

$$\min_z (-f(z)) = -\max_z (f(z)). \quad \otimes$$

Thus:  $-\min_{y \in \bar{A}_2} \max_{x \in \bar{A}_1} u_1(x, y)$

$$\stackrel{\textcircled{2}}{=} \max_{y \in \bar{A}_2} -\max_{x \in \bar{A}_1} u_1(x, y) \quad \text{2SG}$$

$$\stackrel{\textcircled{3}}{=} \max_{y \in \bar{A}_2} \min_{x \in \bar{A}_1} -u_1(x, y) = \max_{y \in \bar{A}_2} \min_{x \in \bar{A}_1} u_2(x, y). \quad \square$$

## Maximinimax Theorem:

- (a) Whenever  $(x^*, y^*)$  is a NE of a 2SG  $G$ , then  $x^*$  and  $y^*$  are maximinimizers (MIs) of player 1 and player 2, resp.
- (b) If  $(x^*, y^*)$  is a NE of a 2SG  $G$ , then
- $$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) = u_1(x^*, y^*);$$
- This means that all NE in  $G$  have the same payoffs.
- (c) If
- $$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y),$$
- and  $x^*$  and  $y^*$  are MIs for players 1 and 2,

then  $(x^*, y^*)$  is a NE.

Corollary: Let  $(x_1^*, y_1^*)$  and  $(x_2^*, y_2^*)$  are NE of ZSG G, then  $(x_1^*, y_2^*)$  and  $(x_2^*, y_1^*)$  are also NE of G.

Proof (corollary):

With (a) we get  $x_1^*, x_2^*$  MN for p. 1 }  
and  $y_1^*, y_2^*$  MN for p. 2 } @

With (b) we get  $\max \min \dots = \min \max \dots$  } (B)

With (a), (B), and (c), we get  $(x_1^*, y_2^*)$  and  $(x_2^*, y_1^*)$  are also NE. □

## Proof of minmax theorem:

(a) Let  $(x^*, y^*)$  be a NE

Use s.t.  $x^*$  is M for player 1

(Proof for  $y^*$  M for player 2 omitted,  
very similar.)

$$(i) \text{ NE} \Rightarrow u_2(x^*, y^*) \geq u_2(x^*, y) \quad \forall y \in A_2$$

$$u_1 = -u_2 \\ \Rightarrow u_1(x^*, y^*) \leq u_1(x^*, y) \quad \forall y \in A_2$$

$$\Rightarrow \boxed{u_1(x^*, y^*) \leq \min_{y \in A_2} u_1(x^*, y)}$$

$$\leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y)$$

$$(ii) \text{ NE} \Rightarrow u_1(x^*, y^*) \geq u_1(x, y^*) \quad \forall x \in \Omega_1$$

$$\Rightarrow u_1(x^*, y^*) \geq \max_{x \in \Omega_1} u_1(x, y^*)$$

$$\Rightarrow u_1(x^*, y^*) \geq \max_{x \in \Omega_1} \min_{y \in \Omega_2} u_1(x, y)$$

$$\oplus 1 \oplus \oplus \Rightarrow u_1(x^*, y^*) = \max_{x \in \Omega_1} \min_{y \in \Omega_2} u_1(x, y) \quad (I)$$

$\Rightarrow x^*$  is a m.n for player 1.