

		2		
		L	R	
1	T	1, -1	2, -2	→ 1
	B	-2, 2	-4, 4	→ -4

	↓	↓		
	-1	-2	-1	→ $\max_{a_1} \min_{a_2} u_1(\dots)$
	1	2	1	→ $\min_{a_2} \max_{a_1} u_1(\dots)$

Lemma: Let G be a ZSG. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x, y).$$

Proof: For each real-valued function f , it holds

$$\min_z (-f(z)) = -\max_z (f(z)). \quad \textcircled{*}$$

Thus: $-\min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$

$$\textcircled{*} = \max_{y \in A_2} -\max_{x \in A_1} u_1(x, y)$$

$$\textcircled{*} = \max_{y \in A_2} \min_{x \in A_1} -u_1(x, y) \stackrel{\text{ZSG}}{=} \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \quad \square$$

Maximinimizer Theorem:

(a) Whenever (x^*, y^*) is a NE of a ZSG G , then x^* and y^* are maximinimizers (MM) of player 1 and player 2, resp.

(b) If (x^*, y^*) is a NE of a ZSG G , then

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y) = u_1(x^*, y^*);$$

This means that all NE in G have the same p-yoffs.

(c) If $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$, and x^* and y^* are MMs for players 1 and 2,

then (x^*, y^*) is a NE.

Corollary: Let (x_1^*, y_1^*) and (x_2^*, y_2^*) are NE of ZSG G , then (x_1^*, y_2^*) and (x_2^*, y_1^*) are also NE of G .

Proof (corollary):

With (a) we get x_1^*, x_2^* MM for p. 1 } $\textcircled{\alpha}$
 and y_1^*, y_2^* MM for p. 2 } $\textcircled{\beta}$

With (b) we get $\max \min \dots = \min \max \dots$ } $\textcircled{\beta}$

With $\textcircled{\alpha}, \textcircled{\beta}$, and (c), we get (x_1^*, y_2^*) and (x_2^*, y_1^*) are also NE. \square

Proof of minmax theorem:

(a) Let (x^*, y^*) be a NE

We show that x^* is MM of player 1.

(proof for y^* MM for player 2 omitted, very similar.)

$$(i) \text{ NE} \Rightarrow u_2(x^*, y^*) \geq u_2(x^*, y) \quad \forall y \in A_2$$

$$\stackrel{u_1 = -u_2}{\Rightarrow} u_1(x^*, y^*) \leq u_1(x^*, y) \quad \forall y \in A_2$$

$$\Rightarrow \boxed{u_1(x^*, y^*) \leq \min_{y \in A_2} u_1(x^*, y)} \quad \oplus$$

$$\leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y)$$

$$(ii) \text{ NE} \Rightarrow u_1(x^*, y^*) \geq u_1(x, y^*) \quad \forall x \in A_1$$

$$\Rightarrow u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*)$$

$$\Rightarrow \boxed{u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y)}$$

$$\oplus \wedge \oplus \Rightarrow \boxed{u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y)} \quad \oplus$$

$\Rightarrow x^*$ is a MM for player 1. (I)