# Sealed-Bid Auctions

An object has to be assigned to one player $i \in \{1, \dots, n\}$ in exchange for a payment. For each player $i$, $v_i$ is the valuation of player $i$ of the object. W.l.o.g., we assume that $v_1 > v_2 > v_3 > \dots > v_n$.

<u>Mechanism:</u> Players simultaneously give their bids $b_1, b_2, \dots, b_n \geq 0$. The object is given to the bidder $i$ with maximal bid $b_i$. Break ties by valuation order, i.e., if $b_i = b_j$ are the highest bids, then $i$ will win iff $i < j$.

**First price auction:** The payment by the winner is the highest bid.

**Second price auction:** The payment by the winner is the highest bid of non-winning bidders.

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**Formalization:**

$$N = \{1, \dots, n\}$$

$$A_i = \{ b_i \mid b_i \in \mathbb{R}_0^+ \}$$

for first-price auction.

$$u_i(b) = \begin{cases} 0 & \text{, if player } i \text{ does not win} \\ v_i - b_i & \text{, otherwise} \end{cases}$$

For second-price auction:

$$u_i(b) = \begin{cases} 0, & \text{if player } i \text{ does not win} \\ v_i - \max b_{-i}, & \text{otherwise.} \end{cases}$$

<u>Example:</u> Three bidders 1, 2, 3.

$v_1 = 100$, $v_2 = 80$, $v_3 = 53$

$b_1 = 90$, $b_2 = 85$, $b_3 = 45$

Bidder 1 wins both types of auctions.

First-price auction: $u_1(b) = v_1 - b_1 = 100 - 90 = 10.$

Second-price auction: $u_1(b) = v_1 - b_2 = 100 - 85 = 15.$

**Proposition:** In a second-price auction, bidding your own valuation, $b_i^+$, is a weakly dominant strategy.

**Proof:** 1) Regardless of what the other bidders do, $b_i^+$ is always a best strategy.

Case I) $i$ wins: $i$ has to pay $\max b_{-i} \leq v_i$, which means that $u_i(b_{-i}, b_i^+) \geq 0$.

Case I.1) $i$ decreases his bid: does not help. (might still win at the same payment, or might lose and get utility 0).

Case I.2) $i$ increases his bid: $i$ still wins, pays the same amount as before.

Case II) $i$ loses. $u_i(b_{-i}, b_i^+) = 0$.

Case II.1: $i$ decreases his bid:

still loses, still utility 0.

Case II.2: $i$ increases his bid:

If $i$ still loses, still utility 0;

if $i$ becomes winner, $i$ pays more than

the object is worth to him $\Rightarrow$ negative utility.

2) $b_i^+$ is <u>strictly</u> better than any other strategy under <u>some</u> opponent profile $b_{-i}$.

Let $b_i'$ be some strategy $\neq b_i^+$.

Case I) $b_i' < b_i^+$. Now let us consider $b_{-i}$ with $b_i^+ > \max b_{-i} > b_i'$. With $b_i'$, we do not win any more, i.e., we have $u_i(b_{-i}, b_i') = 0$, whereas $u_i(b_{-i}, b_i^+) > 0$.

Case II) $b_i' > b_i^+$. Consider $b_i' > \max b_{-i} > b_i^+$. Here $u_i(b_{-i}, b_i') < 0$, but $u_i(b_{-i}, b_i^+) \geq 0$. $\square$

<u>Remark:</u> A profile of weakly dominant strategies is a NE, because for no player there is an incentive to deviate to a different action.

<u>Remark:</u> There are other NE, as well.

# Security : Minimax

| | | Pl. 2 | | |
|---|---|---|---|---|
| | | L | R | |
| Pl. 1 | T | $\boxed{2, 1}$ | 2, −20 | → 2 |
| | M | 3, 0 | −10, 1 | → −10 |
| | B | −100, 2 | 3, 3 | → −100 |
| | | ↓ 0 | ↓ −20 | |

# Zero-Sum Games and NE

<u>Def.:</u> A zero-sum game (ZSG) is a 2-player strategic game

$$G = \langle \underbrace{\{1,2\}}_{N}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$$

such that for all profiles $a \in A$:

$$u_1(a) + u_2(a) = 0.$$

<u>Remark:</u> Can be generalized to constant-sum games, where the utilities sum up to a constant $c$.

Ex:

2

| 1 | L | C | R | |
|---|---|---|---|---|
| T | 8, -8 | 3, -3 | -6, 6 | → -6 |
| M | 2, -2 | -1, 1 | 3, -3 | → -1 |
| B | -6, 6 | 4, -4 | 8, -8 | → -6 |
| | ↓ -8 | ↓ -4 | ↓ -8 | |

Idea: Try to play it safe. Assume that the other player tries to harm you as much as he can.

**Def.:** Let $G$ be a ZSG; $x^* \in A_1$ is called a <u>maximiniminizer</u> for player 1 iff:

$$\boxed{\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y)}$$

for all $x \in A_1$.

Similarly for player 2.

U.d: $u_i(b_i^+, b_{-i}) \geq u_i(b_i', b_{-i}) \quad \forall b_{-i}$

<u>and</u> $u_i(b_i^+, b_{-i}) > u_i(b_i', b_{-i})$ for <u>some</u> $b_{-i}$