Sealed-Bid Auctions

An object has to be assigned to one player $i \in \{1, \dots, n\}$ in exchange for a payment. For each player i, v_i is the valuation of player i of the object. W.l.o.g., we assume that $v_1 > v_2 > v_3 > \dots > v_n$.

Mechanism: Players simultaneously give their bids $b_1, b_2, \dots, b_n \geq 0$. The object is given to the bidder i with maximal bid b_i. Break ties by valuation order, i.e., if $b_i = b_j$ are the highest bids, then i will win iff $i < j$.

First price auction: The payment by the winner is the highest bid.

Second price auction: The payment by the winner is the highest bid of non-winning bidders.

Formalization:

$N = \{1, \dots, n\}$

$A_i = \{ b_i \mid b_i \in \mathbb{R}_0^+ \}$

for first-price auction.

$$u_i(b) = \begin{cases} 0 & \text{, if player } i \text{ does not win} \\ v_i - b_i & \text{, otherwise} \end{cases}$$

For second-price auction:

$$u_i(b) = \begin{cases} 0 & \text{, if player } i \text{ does not win} \\ v_i - \max b_{-i} & \text{, otherwise.} \end{cases}$$

Example: Three bidders 1, 2, 3.

$v_1 = 100$, $v_2 = 80$, $v_3 = 53$

$b_1 = 90$, $b_2 = 85$, $b_3 = 45$

Bidder 1 wins both types of auctions.

First-price auction: $u_1(b) = v_1 - b_1 = 100 - 90 = 10$.

Second-price auction: $u_1(b) = v_1 - b_2 = 100 - 85 = 15$.

Proposition: In a second-price auction, bidding your own valuation, b_i^+, is a weakly dominant strategy.

Proof: 1) Regardless of what the other bidders do, b_i^+ is always a best strategy.

Case I) i wins: i has to pay $\max b_{-i} \leq v_i$, which means that $u_i(b_{-i}, b_i^+) \geq 0$.

Case I.1) i decreases his bid: does not help. (Might still win at the same payment, or might lose and get utility 0).

Case I.2) i increases his bid: i still wins, pays the same amount as before.

Con II) i loses. $u_i(b_{-i}, b_i^+) = 0$.

Case II.1: i decreases his bid:
Still loses, still utility 0.

Case II.2: i increases his bid:
If i still loses, still utility 0;
if i becomes winner, i pays more than the object is worth to him $\Rightarrow$ negative utility.

2) b_i^+ is <u>strictly</u> better than any other strategy under <u>some</u> opponent profile b_{-i}.

Let b_i' be some strategy $\neq b_i^+$.

Con I) $b_i' < b_i^+$. Now let us consider b_{-i} with $b_i^+ > \max b_{-i} > b_i'$. With b_i', we do not win any more, i.e., we have $u_i(b_{-i}, b_i') = 0$, whereas $u_i(b_{-i}, b_i^+) > 0$.

Con II) $b_i' > b_i^+$. Consider $b_i' > \max b_{-i} > b_i^+$. Here $u_i(b_{-i}, b_i') < 0$, but $u_i(b_{-i}, b_i^+) \geq 0$. $\square$

<u>Remark:</u> A profile of weakly dominant strategies is a NE, because for no player there is an incentive to deviate to a different action.

<u>Remark:</u> There are other NE, as well.

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		Pl. 2: L	Pl. 2: R	
Pl. 1	T	$\boxed{2, 1}$	2, −20	→ 2
	M	3, 0	−10, 1	→ −10
	B	−100, 2	3, 3	→ −100
		↓ 0	↓ −20	

Zero-Sum Games and NE

Def.: A zero-sum game (ZSG) is a 2-player strategic game

$$G = \langle \underbrace{\{1,2\}}_{N}, (A_i)_{i\in N}, (u_i)_{i\in N} \rangle$$

such that for all profiles $a \in A$:

$$u_1(a) + u_2(a) = 0.$$

Remark: Can be generalized to constant-sum games, where the utilities sum up to a constant c.

Ex:

		2: L	C	R	
1	T	8, −8	3, −3	−6, 6	→ −6
	M	2, −2	−1, 1	3, −3	→ −1
	B	−6, 6	4, −4	8, −8	→ −6
		↓ −8	↓ −4	↓ −8	

Idea: Try to play it safe. Assume that the other player tries to hurt you as much as he can.

Def.: Let G be a ZSG; $x^* \in A_1$ is called a maximiniser for player 1 iff:

$$\boxed{\min_{y\in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y)}$$

for all $x \in A_1$.

Similarly for player 2.

U.d.: $u_i(b_i^+, b_{-i}) \geq u_i(b_i^!, b_{-i})$ $\forall b_{-i}$

and $u_i(b_i^+, b_{-i}) > u_i(b_i^!, b_{-i})$ for some b_{-i}