

Recall: A strategy  $a'_i \in A_i$  is strictly dominated

by strategy  $a_i^+ \in A_i$  iff  $\forall a_{-i} \in A_{-i}$ :

$$u_i(a_i^+, a_{-i}) > u_i(a'_i, a_{-i}).$$

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Strategy  $a'_i \in A_i$  is weakly dominated by  $a_i^+ \in A_i$

iff  $\forall a_{-i} \in A_{-i} : u_i(a_i^+, a_{-i}) \geq u_i(a'_i, a_{-i})$

and  $\exists a_{-i} \in A_{-i} : u_i(a_i^+, a_{-i}) > u_i(a'_i, a_{-i})$

Def. (Nash equilibrium).

A Nash equilibrium (NE) of a strategic game  $G$  is a strategy profile  $a^* \in A$  such that for all  $i \in N$ :

$$u_i(a^*) \geq u_i(a_{-i}^*, a_i) \quad \text{f.o. } a_i \in A_i.$$

Ex:

	A	B
A	2, 2	0, 0
B	0, 0	1, 1

A red '1' is written to the left of the table. A red '2' is written above the column headers. Red circles highlight the payoffs (2, 2) and (1, 1). Red arrows point from the (1, 1) cell to the (2, 2) cell and from the (0, 0) cell to the (1, 1) cell.

$$u_1(A, A) = 2$$

(A, A) NE.

# Ex. 2: Prisoner's Dilemma

		2	
		C	D
1	C	(1, 1)	(4, 0)
	D	(0, 4)	(3, 3)

A hand-drawn 2x2 payoff matrix for a Prisoner's Dilemma. The rows represent Player 1's strategies (C and D) and the columns represent Player 2's strategies (C and D). The payoffs are written as (Player 1, Player 2). The cells (1,1), (1,0), and (3,3) are circled. Arrows indicate the best response for each player: Player 1's best response to C is D (0,4) and to D is C (4,0); Player 2's best response to C is D (4,0) and to D is C (3,3). The intersection of these best responses is (4,0), which is the Nash equilibrium.

# Ex. 3, Matching Penns

		2	
		H	T
1	H	(1, -1)	(-1, 1)
	T	(-1, 1)	(1, -1)

The image shows a 2x2 payoff matrix for the Matching Penns game. The rows are labeled '1' and the columns are labeled '2'. The strategies for player 1 are 'H' and 'T', and for player 2 are 'H' and 'T'. The payoffs are circled in each cell. Arrows indicate a cycle: from (H,H) to (H,T), from (H,T) to (T,T), from (T,T) to (T,H), and from (T,H) to (H,H).

Alternative Def. of NE:

$$B_i(a_{-i}) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \text{ f.o. } a'_i \in A_i\}$$

$B_i : A_{-i} \rightarrow 2^{A_i}$  is called best-response function.

A NE  $a^*$  is a profile  $a^*$  s.t.

$$a_i^* \in B_i(a_{-i}^*) \text{ f.o. } i \in N.$$

$$\text{I.e. } a^* \in B(a^*) := \prod_{i=1}^{|N|} B_i(a_{-i}^*).$$

# Iterative elimination of weakly dominated strategies:

1

		2	
		L	R
T	(2, 1)	(0, 0)	
M	(2, 1)	(1, 1)	
B	(0, 0)	(1, 1)	

NE: (T, L),  
 (M, L),  
 (M, R),  
 (B, R)

[1] ~~down. by M~~

[2] ~~down. by L~~

1

		2	
		L	R
T	(2, 1)	(0, 0)	
M	(2, 1)	(1, 1)	
B	(0, 0)	(1, 1)	

[1] (B, R)

[2] (B, R)

## Iterative Elimination of strictly dominated strategies.

Lemma: Let  $G$  be a finite strategic game and  $G'$  be the game resulting from eliminating one strictly dominated strategy from  $G$ . Then the NEs of  $G$  are exactly the NEs of  $G'$ .

Proof: Let  $a_i'$  be the eliminated strategy.

Then ex.  $a_i^+$  s.t. f.o.  $a_{-i} \in A_{-i}$ :

$$u_i(a_{-i}, a_i') < u_i(a_{-i}, a_i^+) \quad (1)$$

" $\Rightarrow$ ": Let  $a^*$  be a NE of  $G$ . Then

$$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i'') \quad \text{f.o. } a_i'' \in A_i$$

$$\Rightarrow \underline{u_i(a_{-i}^*, a_i^*)} \geq u_i(a_{-i}^*, a_i^+) \stackrel{(1)}{>} \underline{u_i(a_{-i}^*, a_i')}.$$

$\Rightarrow a_i^* \neq a_i' \Rightarrow$  NE strategy was not eliminated!

$\Rightarrow a^*$  s.t. a NE in  $G'$ .

" $\Leftarrow$ ": Let  $a^*$  be a NE in  $G'$ .

For players  $j \neq i$ :  $a_j^* \in B_j'(a_{-j}^*) = B_j(a_{-j}^*)$ .

(no strategy of player  $j$  was eliminated.)

For player  $i$ :  $u_i(a_{-i}^*, a_i^*) \geq$

$$u_i(a_{-i}^*, a_i^+) >^{(1)}$$

$$u_i(a_{-i}^*, a_i')$$

$a^*$  NE in  $G'$ ,  
 $a_i^+$  in  $G'$ .

$\Rightarrow$   $a_i'$  no better response to  $a_{-i}^*$  than  $a_i^*$  (in  $G$ )

$\Rightarrow a_i^* \in B_i(a_{-i}^*) \Rightarrow a^*$  also NE in  $G$ .  $\square$

Corollary: If IEDS with strict dominance results in a unique strategy profile  $a^*$ , then  $a^*$  is the unique NE of the original game.

Proof: Induction over previous lemma.

Remark: IEDS with strict dom. does not depend on elimination order.