<u>Recall:</u> A strategy $a_i^! \in A_i$ is <u>strictly dominated</u> by strategy $a_i^+ \in A_i$ iff f.a. $a_{-i} \in A_{-i}$:

$$u_i(a_i^+, a_{-i}) > u_i(a_i^!, a_{-i}).$$

Strategy $a_i^! \in A_i$ is <u>weakly dominated</u> by $a_i^+ \in A_i$

iff • f.a. $a_{-i} \in A_{-i}$: $u_i(a_i^+, a_{-i}) \geq u_i(a_i^!, a_{-i})$

and • for some $a_{-i} \in A_{-i}$: $u_i(a_i^+, a_{-i}) > u_i(a_i^!, a_{-i})$

Def. (Nash equilibrium).

A Nash equilibrium (NE) of a strategic game G is a strategy profile $a^* \in A$ such that for all $i \in N$:

$$u_i(a^*) \geq u_i(a^*_{-i}, a_i) \quad \text{f.a.} \quad a_i \in A_i.$$

Ex:

2

	A	B
A	2, 2	0, 0
B	0, 0	1, 1

1

$u_1(A, A) = 2$

(A, A) NE.

Ex. 2: Prisoner's Dilemma

		2: C	2: D
1	C	1,1	4,0
	D	0,4	3,3

Ex. 3: Matching Pennies

		2: H	2: T
1	H	1, −1	−1, 1
	T	−1, 1	1, −1

Alternative Def. of NE:

$$B_i(a_{-i}) = \{ a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ f.a. } a_i' \in A_i \}$$

$B_i : A_{-i} \to 2^{A_i}$ is called <u>best-response function</u>.

A NE a^* is a profile a^* s.t.

$a_i^* \in B_i(a_{-i}^*)$ f.a. $i \in N$.

I.e. $a^* \in B(a^*) := \prod_{i=1}^{|N|} B_i(a_{-i}^*)$.

Iterative elimination of weakly dominated strategies:

2

	L	R
T	2,1	0,0
M	2,1	1,1
~~B~~	0,0	1,1

1

$\boxed{1}$ dom. by M

$\boxed{2}$ dom. by L

NE: (T, L),
(M, L),
(M, R)
(B, R)

2

	L	R
~~T~~	2,1	0,0
M	2,1	1,1
~~B~~	0,0	1,1

1

$\boxed{1}$ (by R)

$\boxed{2}$ (by R)

Iterative Elimination of <u>strictly</u> dominated strategies.

<u>Lemma:</u> Let G be a finite strategic game and G' be the game resulting from eliminating <u>one</u> strictly dominated strategy from G. Then the NEs of G are exactly the NEs of G'.

<u>Proof:</u> Let a_i' be the eliminated strategy.

Then ex. a_i^+ s.t. f.a. $a_{-i} \in A_{-i}$:

$$u_i(a_{-i}, a_i') < u_i(a_{-i}, a_i^+) \qquad (1)$$

<u>"$\Rightarrow$"</u>: Let a^* be a NE of G. Then

$$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i'') \quad \text{f.a. } a_i'' \in A_i$$

$$\Rightarrow \underline{u_i(a_{-i}^*, a_i^*)} \geq u_i(a_{-i}^*, a_i^+) \overset{(1)}{>} \underline{u_i(a_{-i}^*, a_i')}.$$

$\Rightarrow a_i^* \neq a_i'$ $\Rightarrow$ NE strategy was not eliminated!

$\Rightarrow a^*$ still NE in G'.

"$\Leftarrow$": Let a^* be a NE in G'.

For players $j \neq i$: $a_j^* \in B_j'(a_{-j}^*) = B_j(a_{-j}^*)$.

(no strategy of player j was eliminated.)

For player i: $u_i(a_{-i}^*, a_i^*) \geq$ a^* NE in G', a_i^+ in G'.

$$u_i(a_{-i}^*, a_i^+) \overset{(1)}{>}$$

$$u_i(a_{-i}^*, a_i')$$

$\Rightarrow$ a_i' no better response to a_{-i}^* than a_i^* (in G)

$\Rightarrow$ $a_i^* \in B_i(a_{-i}^*)$ $\Rightarrow$ a^* also NE in G. $\square$

Corollary: If IEDS with <u>strict</u> dominance results in a <u>unique</u> strategy profile a^*, then a^* is the unique NE of the original game.

Proof: Induction over previous lemma.

Remark: IEDS with <u>strict</u> dom. does <u>not</u> depend on elimination order.