

Recall: A strategy $a_i' \in A_i$ is strictly dominated by strategy $a_i^* \in A_i$ iff f.a. $a_{-i} \in A_{-i}$:
 $u_i(a_i^*, a_{-i}) > u_i(a_i', a_{-i})$.

Strategy $a_i' \in A_i$ is weakly dominated by $a_i^* \in A_i$ iff
 • f.a. $a_{-i} \in A_{-i}$: $u_i(a_i^*, a_{-i}) \geq u_i(a_i', a_{-i})$
 and • for some $a_{-i} \in A_{-i}$: $u_i(a_i^*, a_{-i}) > u_i(a_i', a_{-i})$

Ex. 2: Prisoner's Dilemma

	1	2
1	C	D
D	0, 4	3, 3

Def. (Nash equilibrium).

A Nash equilibrium (NE) of a strategic game G is a strategy profile $a^* \in A$ such that for all $i \in N$:

$$u_i(a^*) \geq u_i(a_{-i}^*, a_i) \quad \text{f.a. } a_i \in A_i.$$

Ex:

	1	2
1	A	2, 2
2	0, 0	1, 1

$$\begin{aligned} u_1(A, A) &= 2 \\ (A, A) &\text{ NE.} \end{aligned}$$

Ex. 3: Matching Pennies

	1	2
H	H	-1, -1
T	1, 1	1, -1

Alternative Def. of NE:

$$\mathcal{B}_i(a_{-i}) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \text{ f.o. } a'_i \in A_i\}$$

$\mathcal{B}_i : A_{-i} \rightarrow 2^{A_i}$ is called best-response function.

A NE a^* is a profile a^* s.t.

$$a_i^* \in \mathcal{B}_i(a_{-i}^*) \text{ f.o. } i \in N.$$

$$\text{i.e. } a^* \in \mathcal{B}(a^*) := \prod_{i=1}^{|N|} \mathcal{B}_i(a_{-i}^*).$$

Iterative elimination of weakly dominated strategies:

NE: (T, L) , (M, L) , (M, R) , (B, R)

① down. by M

② down. by L

	L	R
T	2, 1	0, 0
M	2, 1	1, 1
B	0, 0	1, 1

③ (B, R)

Iterative Elimination of strictly dominated strategies:

Lemma: Let G be a finite strategic game and G' be the game resulting from eliminating one strictly dominated strategy from G . Then the NEs of G are exactly the NEs of G' .

Proof: Let a_i' be the eliminated strategy.

Then ex. a_i^+ s.t. f.o. $a_{-i} \in A_{-i}$:

$$u_i(a_{-i}, a_i') < u_i(a_{-i}, a_i^+) \quad (1)$$

\Rightarrow : Let a^* be a NE of G . Then

$$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i'') \text{ f.o. } a_i'' \in A_i$$

$$\Rightarrow u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+) \stackrel{(1)}{>} u_i(a_{-i}^*, a_i')$$

$$\Rightarrow a_i^* \neq a_i' \Rightarrow \text{NE strategy was not eliminated!}$$

$$\Rightarrow a^* \text{ s.t. NE in } G'.$$

\Leftarrow : Let a^* be a NE in G' .

For player $j \neq i$: $a_j^* \in B_j'(a_{-j}^*) = B_j(a_{-j}^*)$.

(no strategy of player j was eliminated.)

For player i : $u_i(a_{-i}^*, a_i^*) > \begin{matrix} a_i^* \text{ NE in } G' \\ a_i^* \text{ in } G' \end{matrix}$

$$u_i(a_{-i}^*, a_i^+) >$$

$$u_i(a_{-i}^*, a_i')$$

$\Rightarrow a_i^+$ no better response to a_{-i}^* than a_i^* (in G)

$\Rightarrow a_i^* \in B_i(a_{-i}^*) \Rightarrow a^*$ also NE in G . \square

Corollary: If IEDS with strict

dominance results in a unique strategy profile a^* , then a^* is the unique NE of the original game.

Proof: Induction over previous lemma.

Remark: IEDS with strict dom. does not depend on elimination order.