

Axioms for Voting Systems

Neutrality

"Names" of candidates should not be relevant.

Anonymity

"Names" of voters should not be relevant.

Majority criterion

A candidate, who is ranked highest by the majority ($> 50\%$), should be the winner.

Plurality +
Borda -

Mutual majority c.

If a majority of the voters prefer candidates from one subset over all other candidates, then the winner should be in this subset.

Plurality, Borda -
IRV +

P1: 30% A, 30% B, 40% C

1

Independence of Irrelevant Alternatives (IIA)

The outcome never changes if a non-winning candidate is added or removed.

- all the VP violate IIA!

Monotonicity c.

If a candidate wins in an election, then he will win, if one voter ranks him higher.

IRV, Runoff -
All others +

Consistency c.

If there are two sets of voters, with separate elections that have the same winner, then the combined election should have the same winner.

Borda +
IRV -

2

Independence of Clone Alternatives

A set of clones is a subset of candidates such that all other candidates are ranked higher or lower than all candidates from S by all voters.

Schulze, IRV +
Borda -

3

4

Polynomial time c.

The winner should be computable in time polynomial in the number of candidates and linear in the number of voters.

All the ones we have seen +

Dodgson -

Resolvability

- 1) For every tie between winners, one possible vote should resolve the tie
- 2) The proportion of preference profiles leading to a tie should approach zero when the number of voters approaches infinity.

All the ones we have seen +

Theorem (May, 1958)

A voting method for two alternatives satisfies anonymity, neutrality, and monotonicity iff it is the plurality method.

Proof: \Leftarrow : obvious!

\Rightarrow : For simplicity, we assume that the number of voters is odd.

Anonymity + neutrality \rightarrow only the number of votes for a candidate count.

Let A be the set of voters preferring candidate a and B the set preferring b .

5

Case 1)

Our voting method selects a if $|A| = |B| + 1$. By monotonicity, a wins whenever $|A| > |B|$, i.e. plurality method.

Case 2)

Our method selects b if $|A| = |B| + 1$. One voter for a changes his preference to b . By monotonicity, b should still be the winner. $|A'| + 1 = |B'|$. This situation is completely

symmetric to the original one, and by neutrality a should win! Contradiction!

□

6

For three or more alternatives, there are no voting methods that satisfy a small set of reasonable criteria.

Arrow's Impossibility Theorem

Desirable properties for social welfare functions:

Def (Total unanimity)

For all $\prec \in L$: $F(\prec, \prec, \dots, \prec) = \prec$

* Def (Partial Unanimity, weak Pareto)

For all $\prec_1, \dots, \prec_n, \prec \in L$ with $F(\prec_1, \dots, \prec_n) = \prec$,

7

8

if $a \prec_i b$ for all $i \in N$, then $a \prec b$.

Remark: Partial unanimity implies total unanimity, but not vice versa.

Def (Independence of Irrelevant Alternatives, IIA):
 $a \prec b$ should only depend on preferences of the voters between a and b , i.e. for all $\prec_1, \dots, \prec_n \in L$ the following should hold: if $\prec = F(\prec_1, \dots, \prec_n)$ and $\prec' = F(\prec'_1, \dots, \prec'_n)$ and $a \prec_i b$ iff $a \prec'_i b$ for all $i = 1, \dots, n$, then this implies that $a \prec b$ iff $a \prec' b$, for $\prec'_1, \dots, \prec'_n \in L$.

9

Def (Non-dictatorship)

A voter i is called dictator for F , if $F(\prec_1, \dots, \prec_i, \dots, \prec_n) = \underline{\prec_i}$ for all orders

$\prec_1, \dots, \prec_n \in L$. F is called non-dictatorial if there is no dictator.

Theorem (Arrow)

Every social welfare function over more than two alternatives that satisfies unanimity and IIA is necessarily dictatorial.

10

Lemma Total unanimity and independence of irrelevant alternatives imply partial unanimity.

Proof: Let $\prec_1, \dots, \prec_n \in L$ with $a \prec_i b$ for all voters i . Let $\prec = F(\prec_1, \dots, \prec_n)$. Consider the following $\prec'_1, \dots, \prec'_n$ with $\prec'_i = \prec_i$ for all i , with total unanimity we set

$$\prec' = F(\prec'_1, \dots, \prec'_n) = F(\prec_1, \dots, \prec_n) = \underline{\prec_1}.$$

$a \prec' b$. Since $a \prec_i b$ iff $a \prec'_i b$ with IIA, we get $a \prec b$ iff $a \prec' b$. Because we know that $a \prec' b$, also $a \prec b$ must hold. \square

Lemma (pairwise neutrality)

Assume F satisfies unanimity (total or partial) and IIA. Let \prec_1, \dots, \prec_n and $\prec'_1, \dots, \prec'_n$ be preference profiles with $a \prec_i b$ iff $a \prec'_i b$ for all $i = 1, \dots, n$. Then $a \prec b$ iff $a \prec' b$ with $\prec = F(\prec_1, \dots, \prec_n)$ and $\prec' = F(\prec'_1, \dots, \prec'_n)$.

Proof: Three cases

1) $a \neq b'$ and $a' \neq b$.

Suppose $a \prec b$.

We construct a new profile $\prec''_1, \dots, \prec''_n$ where $a' \prec''_i a$ (if $a \neq a'$) and $b \prec''_i b'$ (if $b \neq b'$)

11

12

$$\begin{array}{c} \prec_i \\ \left(\begin{array}{c} a \\ b \end{array} \right) \end{array} \quad \begin{array}{c} \prec'_i \\ \left(\begin{array}{c} a' \\ b' \end{array} \right) \end{array} \quad \begin{array}{c} \prec'' \\ \begin{array}{c} a \\ | \\ a' \\ | \\ b \\ | \\ b' \end{array} \end{array}$$

and the order between a and b and a' and b'

should be the same as in \prec_i and \prec'_i
respectively. By IIA $[a \prec'' b]$ for $\prec'' = F(\prec_i, \dots, \prec_n)$. Unanimity implies $a' \prec'' a$ and $b \prec'' b'$.

Now by transitivity, we get $[a' \prec'' b']$.

By III, we have $a' \prec' b'$. Symmetrically,
we can prove the other direction.