Social Choice Theory

- Aggregation of preferences of group members
- Voting and voting procedures
  - Elections
  - Committee decisions
  - European Song Contest

Def (Social Welfare and Social Choice Functions)

Let A be a set of alternatives (candidates) and L be a set of linear orders on A. For n voters, $F: L^n \to L$ denotes a <u>social welfare function</u> and

$f: L^n \to A$ a <u>social choice function</u>.

<u>Notation</u>: A linear order $\prec \in L$ is called a preference relation. For voter i, $\prec_i$.

For example: $a \prec_i b$ means that voter i prefers candidate b over candidate a.

<u>Example</u>:

Assume three voters: 1, 2, 3

and three alternatives a, b, c

1	2	3	4
a	b	c	a
b	c	a	c
c	a	b	b

$b \prec_1 a$, $c \prec_1 b$, $c \prec_1 a$

$c \prec_2 b$ ...

Voting protocols

- Plurality (aka first-past-the-post or winner-takes-it-all):
  - only top preferences are taken into account
  - candidate with most top preferences wins

  Drawback: wasted votes, winner might be preferred only by a minority.
- Plurality with runoff
  - First round: two candidates with most top votes proceed to second round (unless absolute majority)
  - Second round: runoff

  Drawback: takes more time, tactical voting is poss.

Instant runoff voting (transferable votes):

- each voter submits a preference order
- iteratively candidates with the fewest top preferences are eliminated until one candidate remains.

Drawback: Comprise candidates might get not elected.

Borda count:

- each voter submits his preferences order over m candidates
- if a candidate is in position j of a voter's list, he gets m-j points from that voter
- points from all voters are added
- candidate with most points win.

Condorcet winner:

- each voter submits his preference order
- perform a pairwise comparison between candidates
- if one one candidate wins all pairwise comparisons, he is the Condorcet winner

Drawback: Condorcet winner does not always exist.

23 voters, candidates: a, b, c, d, e

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Plurality voting: e

23 voters, candidates: a, b, c, d, e

# voters	8	6	4	3	1	1
1st	e	a	~~b~~	~~c~~	~~d~~	~~d~~
2nd	d	b	~~c~~	~~b~~	~~c~~	~~c~~
3rd	b	c	~~d~~	~~d~~	a	~~b~~
4th	c	e	a	a	~~b~~	e
5th	a	d	e	e	e	a

Plurality voting with runoff

1st round: e and a

2nd round: $6 \times a + 4 \times a + 3 \times a + 1 \times a = 14 \times a$

$8 \times e + 1 \times e = 9 \times e$

↪ a is the winner

23 voters, candidates: a, b, c, d, e

# votes	8	6	4	3	1	1
1st	e	~~a~~	~~b~~	c	d	d
2nd	~~d~~	~~b~~	c	~~b~~	c	c
3rd	~~b~~	c	~~d~~	~~d~~	~~a~~	~~b~~
4th	c	e	~~a~~	~~a~~	~~b~~	e
5th	~~a~~	~~d~~	e	e	e	~~a~~

2x d

8x e

15x c

Instant runoff voting:

1st round elimination: d

2nd round " : b

3rd round " : a

Now c has the majority

23 voters, candidates: a, b, c, d, e

# votes	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

16
18
16
9
1
2

62

b is the wine

Borda count:

a: $8 \cdot 0 + 6 \cdot 4 + 4 \cdot 1 + 3 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 = 33$

b: $8 \cdot 2 + 6 \cdot 3 + 4 \cdot 4 + 3 \cdot 3 + 1 \cdot 1 + 1 \cdot 2 = 62$

c: $8 \cdot 1 + 6 \cdot 2 + 4 \cdot 3 + 3 \cdot 4 + 1 \cdot 3 + 1 \cdot 3 = 50$

d: $8 \cdot 3 + 6 \cdot 0 + 4 \cdot 2 + 3 \cdot 2 + 1 \cdot 4 + 1 \cdot 4 = 46$

e: $8 \cdot 4 + 6 \cdot 1 + 4 \cdot 0 + 3 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 = 39$

23 voters, candidates: a, b, c, d, e

# votes	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Comparison table

	a	b	c	d	e
a	—	7	6	6	14
b	16	—	18	13	15
c	17	5	—	13	15
d	17	10	10	—	9
e	9	8	8	14	—

Condorcet winner: b

- Different winners for different voting protocols
- Which voting protocol prefer?
- Can be used strategically!

Condorcet Paradox

Example of voters 1, 2, 3 on candidates a, b, c

$a \prec_1 b \prec_1 c$

$b \prec_2 c \prec_2 a$

$c \prec_3 a \prec_3 b$

	a	b	c
a	—	1	2
b	2	—	1
c	1	2	—

$a \prec c \qquad c \prec b \qquad b \prec a$

cyclic order

Def A Condorcet method returns a Condorcet winner, if one exist. Otherwise it computes some winner.

One particular Condorcet method:

Schulze method

Relatively new (1997)

Already used by many: Debian, Ubuntu, Pirate Party.

Schulze Method:

Notation: $d(X, Y)$ = number of pairwise comparisons won by X over Y.

Def

For candidates X and Y, there exists a path $C_1, \dots, C_n$ between cd X and Y of strength z if

- $C_1 \equiv X$
- $C_n \equiv Y$
- $d(C_i, C_{i+1}) > d(C_{i+1}, C_i)$ for all $i = 1, \dots, n-1$
- $d(C_i, C_{i+1}) \geq z$ for all $i = 1, \dots, n-1$ and there exist a $j \in \{1, \dots, n-1\}$ s.t. $d(C_j, C_{j+1}) = z$.

Def

Let $p(X, Y)$ be the maximal value z such that there exist a path of strength z from X to Y, and $p(X, Y) = 0$ if no such path exists.

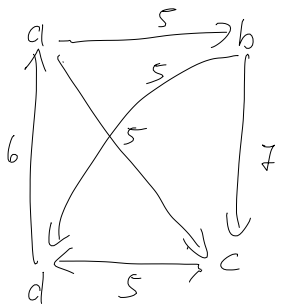
Then the Schulze winner is the Condorcet winner, if it exists. Otherwise, a potential winner is a candidate a such that $\boxed{p(a, X) \geq p(X, a)}$ for all $X \neq a$.

Tie-Breaking is used between potential winners.

# voters	3	2	2	2
1st	a	d	d	c
2nd	b	a	b	b
3rd	c	b	c	d
4th	d	c	a	a

Condorcet winner?

→ no



$d(X,Y)$

	a	b	c	d
a	—	5	5	3
b	4	—	7	5
c	4	2	—	5
d	6	4	4	—

$p(X,Y)$

	a	b	c	d
~~a~~	—	5	5	5
b	5	—	7	5
~~c~~	5	5	—	5
d	6	5	5	—

Potential winners:
b, d