Extensive Games

action & strategies

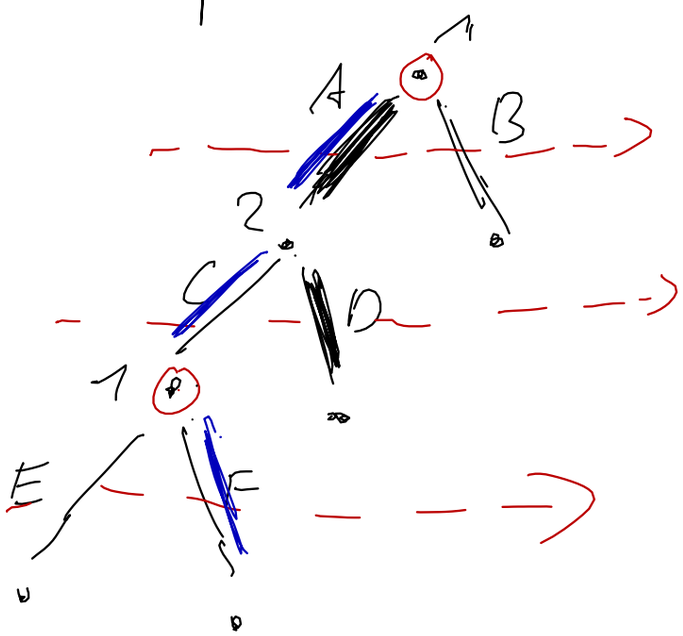
<u>Def</u> (Strategies)

Let $\Gamma = \langle N, A, H, P, (u_i) \rangle$ be a EGWPI. The set of actions $\underline{(h,a)} \in H$ are denoted by $A(h)$. Then a <u>strategy</u> of player i is a function s_i that assigns to each non-terminal history $h \in H \setminus Z$ with $P(h) = i$ an action $a \in A(h)$. The set of strategies of player i is denoted S_i.

<u>Remark</u>: Strategies require us to assign actions to histories even if they are never played.

<u>Notation</u>: Strategies are often described by writing down the actions going through the game tree in breadth-first-search order (i.e., level-by-level, left to right).

Example



Strategies for player 1:

AE, AF, BE, BF

Strategies for player 2:

C, D

$$AE \cong \{ \emptyset \mapsto A,\ (A, C) \mapsto E \}$$

<u>Def</u> (outcome)

The <u>outcome</u> of a strategy profile $s = (s_i)_{i \in \mathcal{N}}$ is the history $h^s = (a_k)_{k=1}^{K}$ such that for all $0 \leq k \leq K$, $K \in \mathbb{N} \cup \{\infty\}$, where $s_{P(a_1, \dots, a_k)}(a_1, \dots, a_k) = a_{k+1}$

The outcome of s is denoted by $O(s)$.

<u>Example</u>

$$O((AF, D)) = (A, D)$$

$$O((AF, C)) = (A, C, F)$$

Nash Equilibrium in Extensive Games

<u>Def</u> (NE)

A Nash equilibrium of an extensive game with perfect information Γ is a strategy profile $s^* = (s_i^*)_{i \in N}$ such that for each player $i \in N$:

$$u_i(O(s^*)) \geq u_i(O(s_{-i}^*, s_i)) \text{ for all } s_i \in S_i.$$

Proposition

The NE of an EGWPI Γ are exactly the NE of the strategic game induced by Γ (called its strategic form), which is defined by

$$G' = \langle N, (A_i')_{i \in N}, (u_i')_{i \in N} \rangle \text{ with}$$

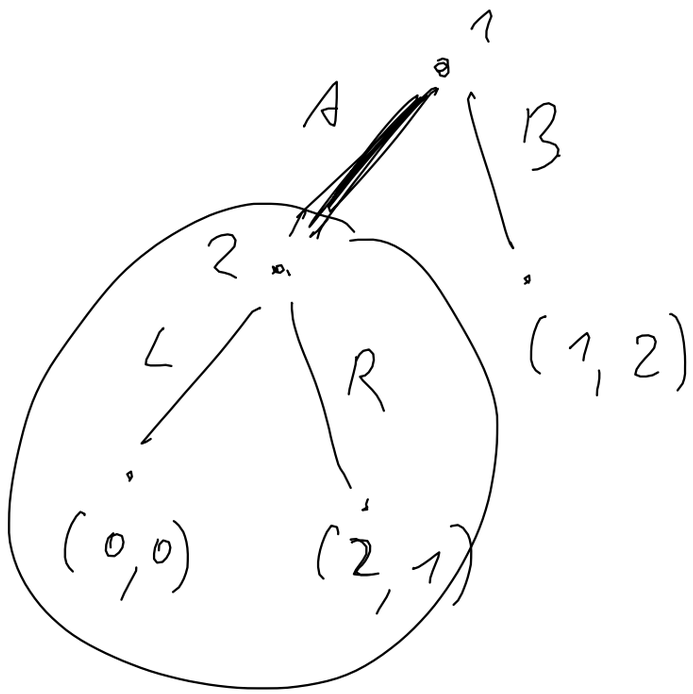
$$A_i' := S_i$$

$$u_i'(a) = u_i(O(s_i)).$$

Remarks

1) Each EGPWI can be transformed into a strategic game, but this can lead exponential blowup of the game representation.

2) The other direction does not hold, because we do not have simultaneous moves in extensive games (yet).



strategic form

$\rightsquigarrow$

	L	R
A	(0,0)	(2,1)
B	(1,2)	(1,2)

BL looks funny

Chosing B as player 1 is only plausible if he fears that player 2 might actually play L. But if player 1 chooses A, player 2 would never play L!

For this reason, L is called a <u>non-credible threat</u>.

Subgame - perfect Equilibria

Let $\Gamma = \langle N, A, H, P, (u_i) \rangle$ be an EGWPI.

<u>Def</u> (Subgame)

The subgame of Γ rooted at history h is the EGWPI $\Gamma(h) = \langle N, A, H|_h, P|_h, (u_i|_h) \rangle$, where:

$$H|_h := \{ h' : (h, h') \in H \}$$

$$P|_h := P((h, h'))$$

$$u_i|_h := u_i((h, h')) \quad \text{for all } (h, h') \in Z$$

For each strategy s_i in Γ, let $s_i|_h(h') := s_i((h, h'))$ be the induced strategy in $\Gamma(h)$.

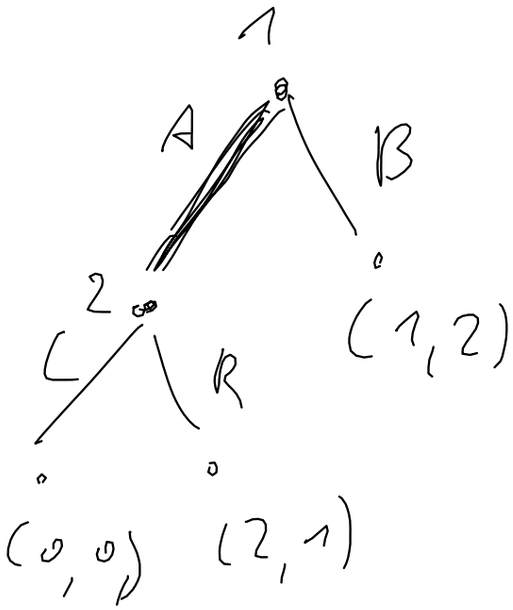
The outcome function of $\Gamma(h)$ is denoted by O_h.

<u>Def</u> (SPE)

A <u>subgame-perfect equilibrium</u> (SPE) of a EGWPI Γ is a strategy profile $s^* = (s_i^*)_{i \in N}$ such that for each history $h \in H$:

$$s^*|_h := (s_i^*|_h)_{i \in N}$$

is a NE of $\Gamma(h)$.



$s = (A, R)$:

$$s = (\{\emptyset \mapsto A\}, \{(A) \mapsto R\})$$

$$s|_{(A)} = (\{\emptyset \mapsto A\}|_{(A)}, \{(A) \mapsto R\}|_{(A)})$$

$$= (\text{—}, \{\emptyset \mapsto R\})$$

$$H = \{\emptyset, (A), (B), (A,L), (A,R)\}$$

$$H|_{(A)} = \{\emptyset, (L), (R)\}$$

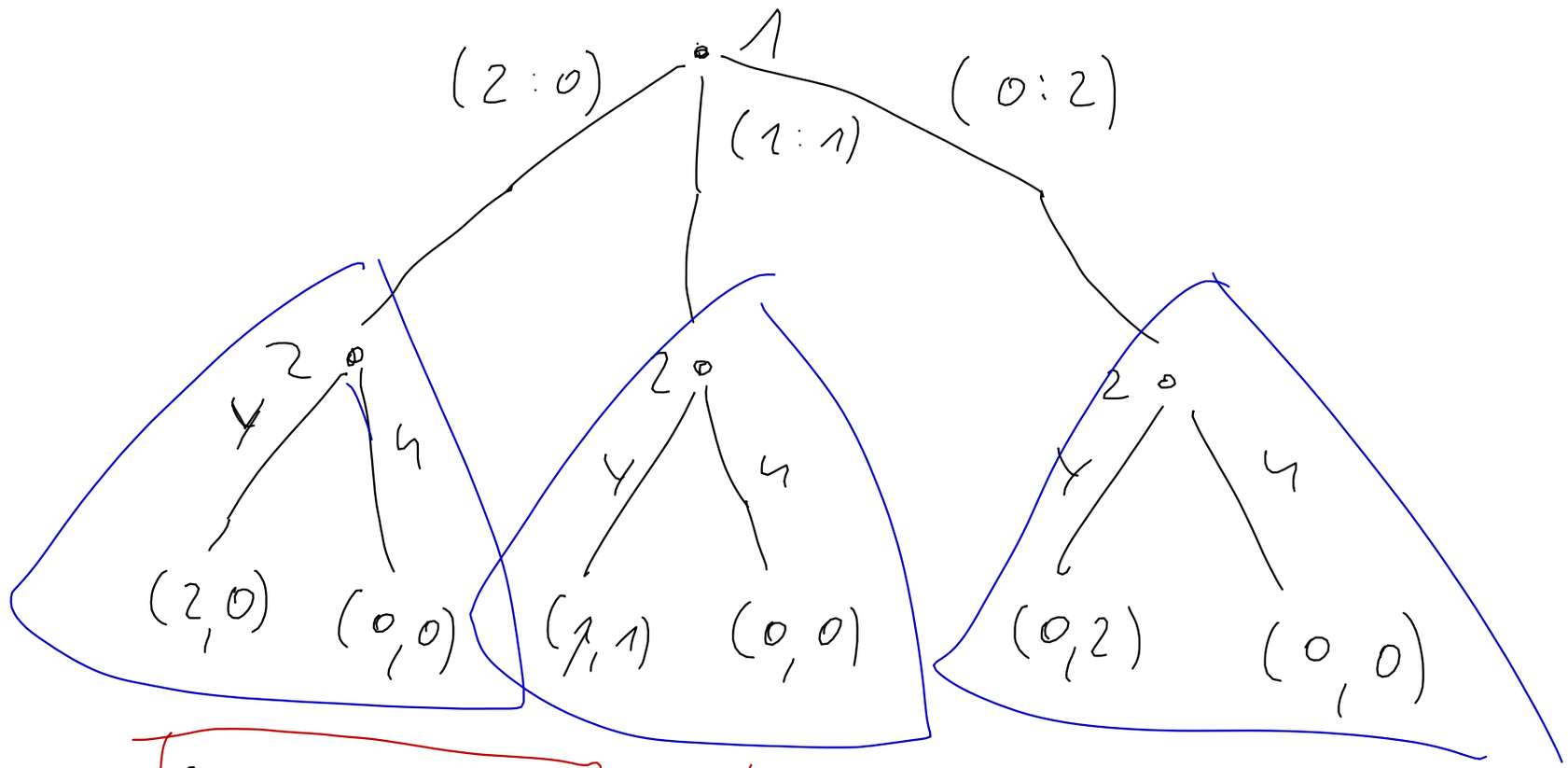
$\Rightarrow s = (A, R)$

$$\left.\begin{array}{ll} h = \emptyset & s \text{ is a NE} \\ h = (A) & s|_{(A)} \text{ is a NE} \end{array}\right\} \Rightarrow \text{SPE}$$

$\Rightarrow s = (B, L)$

$$\left.\begin{array}{ll} h = \emptyset & s \text{ is a NE} \\ h = (A) & s|_{(A)} \text{ is not NE} \end{array}\right\} \Rightarrow \text{not SPE}$$

Example (Sharing game)



Poss profiles: $\boxed{(2:0, yyy)}$ ✓ $\boxed{(1:1, nyy)}$

(2:0, yyn)

(2:0, yny)

(2:0, ynn)

$\boxed{(2:0, nyy)}$

(2:0, nny)

(2:0, nnn)

$\boxed{(2:0, nyn)}$

SPE

(1:1, nyn)

(0:2, nny)

not NE

Questions

- Does an SPE always exist?
- Under which conditions?
- How to compute it?
- what is the complexity?

We show

- It is easy to verify that a profile is an SPE. → "one deviation property" (for finite horizon games)
- For finite games, we can easily compute the SPE by backward induction (Kuhn's Theorem)

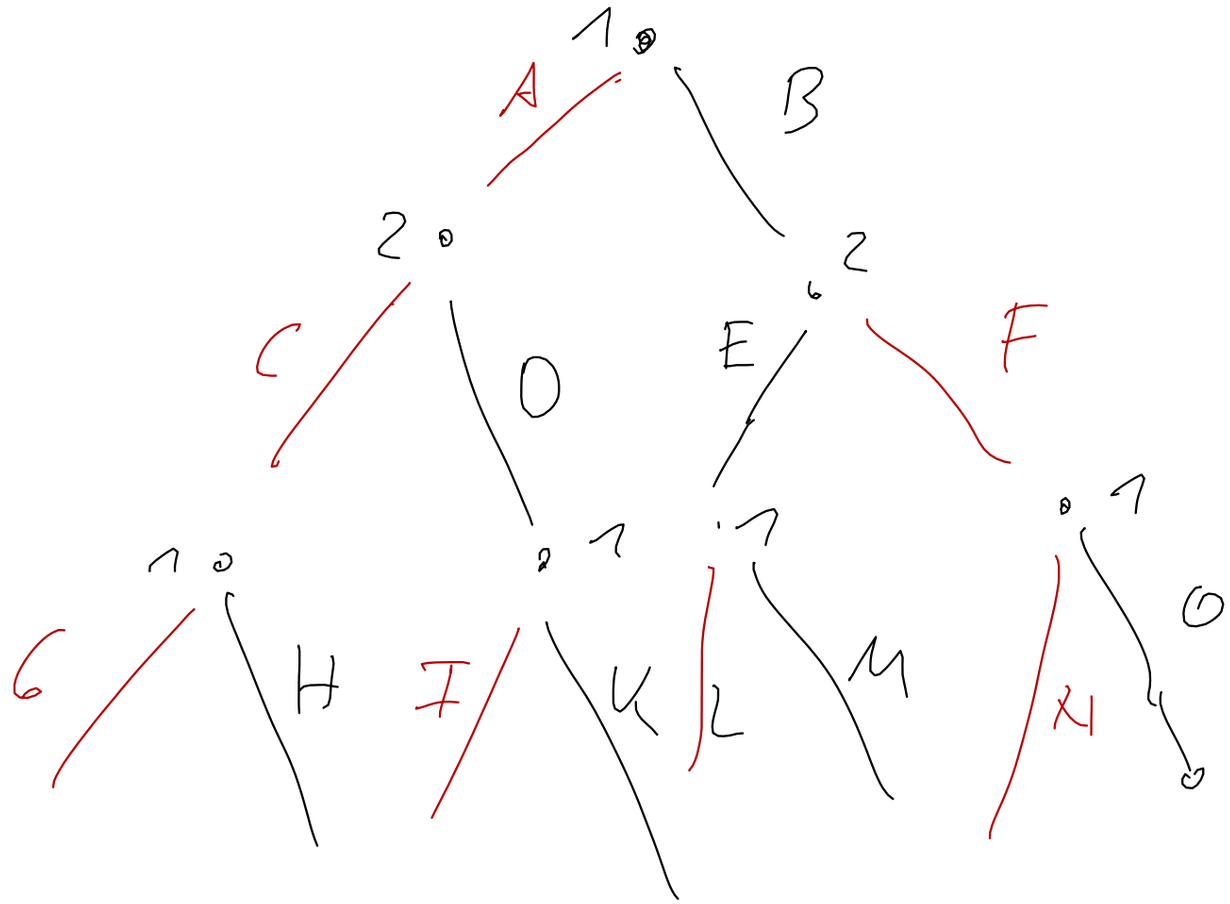
Notation: If Γ is a EGWPI then $\ell(\Gamma)$ denotes the length of the longest history in Γ.

Lemma (One Deviation Property)

Let $\Gamma = \langle N, A, H, P, (u_i) \rangle$ be a finite-horizon EGWPI. Then a strategy profile s^* is an SPE of Γ if and only if for every player $i \in N$ and every history $h \in H$ for which $P(h) = i$, we have

$$u_i|_h(O_h(s^*_{-i}|_h, s_i^*|_h)) \geq u_i|_h(O_h(s^*_{-i}|_h, s_i))$$

for every strategy s_i of player i in the subgame $\Gamma(h)$ that differs from $s_i^*|_h$ only in the action after the initial history of $\Gamma(h)$.


1
A
B
2
2
C
D
E
F
1
1
1
1
G
H
I
K
L
M
N
O