Theorem: NASH is PPAD complete. □

Some further results: Given a finite two-player game G, it is NP-hard to decide whether there exists a MSNE (α, β) in G that has one of the following properties:

(a) player 1 (or 2) receives a payoff $\geq k$.

(b) $\mathcal{U}_1(\alpha,\beta) + \mathcal{U}_2(\alpha,\beta) \geq k$

(c) (α,β) is Pareto optimal, i.e. there is no strategy profile (α',β') such that $\mathcal{U}_i(\alpha',\beta') \geq \mathcal{U}_i(\alpha,\beta)$ for both $i \in \{1,2\}$, and $\mathcal{U}_i(\alpha',\beta') > \mathcal{U}_i(\alpha,\beta)$ for some $i \in \{1,2\}$.

(d) player 1 (or player 2) plays some given a with probability > 0. □

Extensive Games

So far: only simultaneous, one-shot games

Q: How to model sequential structure of many games (chess, ...)?

A: Use <u>extensive game</u> ($\approx$ game trees).

There, players have several choice points where they can decide how to play. <u>Strategies</u> then map choice points to applicable actions.

Definition: An <u>extensive game with perfect information</u> (EGWPI) is a tuple $\Gamma = \langle N, A, H, P, (u_i)_{i \in N} \rangle$ where:

- N is a finite, nonempty set of <u>players</u>.
- A is a finite, nonempty set of <u>actions</u>.
- H is a set of (finite or infinite) sequences over A (called <u>histories</u>) such that:
  - * the empty sequence $\langle\rangle \in H$,
  - * if $\langle a^k \rangle_{k=1}^{K} \in H$ for some $K \in \mathbb{N} \cup \{\infty\}$ and $L < K$ then $\langle a^k \rangle_{k=1}^{L} \in H$ ("prefix closedness")
  - * if $\langle a^k \rangle_{k=1}^{\infty}$ is an action sequence such that $\langle a^k \rangle_{k=1}^{L} \in H$ for all $L \in \mathbb{N}$, then $\langle a^k \rangle_{k=1}^{\infty} \in H$.

  A history is called <u>terminal</u> if it is infinite or if it is not a prefix of any longer history. The set of terminal histories is denoted by Z.
- $P: H \setminus Z \to N$ is the <u>player function</u> assigning to each nonterminal history $h \in H \setminus Z$ a player $P(h)$ whose turn it is to move in h.
- For each player $i \in N$, $u_i: Z \to \mathbb{R}$ is his <u>utility function</u>.

<u>Terminology</u>:
- Γ is <u>finite</u> if H is finite.
- Γ <u>has a finite horizon</u> if H contains no infinite history.

Example (sharing game):

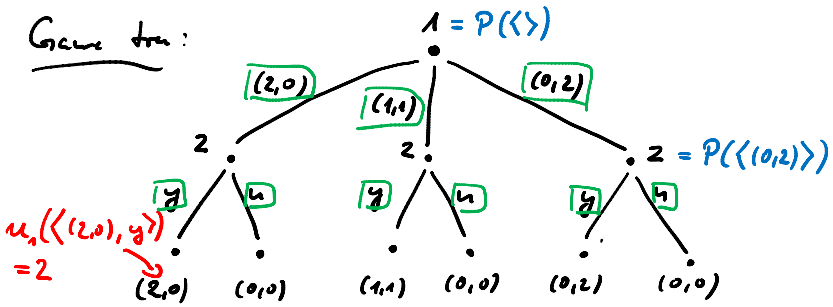
Task: Two players have to share two identical objects.

- Player 1 proposes an allocation.
- Player 2 accepts or declines

objects are allocated as proposed

nobody gets anything

Game tree:



Formally: $\Gamma = \langle N, A, H, P, (u_i)_{i \in N} \rangle$ then

$N = \{1,2\}$, $A = \{(2,0), (1,1), (0,2), y, n\}$

$H = \{\langle\rangle, \langle(2,0)\rangle, \langle(1,1)\rangle, \langle(0,2)\rangle,$
$\langle(2,0), y\rangle, \langle(2,0), n\rangle, \ldots\}$

4 more

$P(\langle\rangle) = 1$, $P(h) = 2$ for all $h \in H \setminus (Z \cup \{\langle\rangle\})$

$u_1(\langle(2,0), y\rangle) = 2$, etc. . .