

Theorem (Nash): Every finite strategic game has a MSNE.

Proof: Apply Kakutani's theorem using

$$X = \mathcal{A} = \overline{\prod_{i \in N} \Delta(A_i)} \quad \text{and} \quad f = B.$$

Need to show: (i) $\mathcal{A} \neq \emptyset$, (ii) \mathcal{A} compact
(i.e. bounded, closed)
(iii) \mathcal{A} convex, (iv) $B(\alpha) \neq \emptyset$ f.e. α
(v) $B(\alpha)$ convex f.e. α
(vi) $\text{Graph}(B)$ is closed.

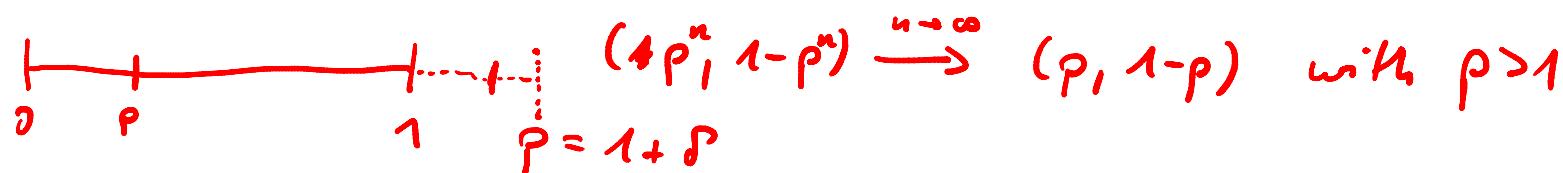
Let $M := \sum_{i \in N} |A_i|$, wlog, $N = \{1, \dots, n\}$.

(i) $A \neq \emptyset$: e.g. $(\underbrace{1, 0, \dots, 0}_{|A_1|-1 \text{ times}}, \underbrace{1, 0, \dots, 0}_{|A_2|-1 \text{ times}}, \dots, 1, 0, \dots, 0)$

(ii) A compact: A bounded by $[0,1]^M$.

A closed: let a_1, a_2, \dots be a sequence in A that converges to $\lim_{n \rightarrow \infty} a_n = a$. Suppose $a \notin A$.

Then either there exists some component a_n s.t. it is < 0 or > 1 , or the components for some player i add up to a value $\neq 1$.



But since a is a limit, the same must hold for some a_n in the sequence. $\Rightarrow a_n \notin A$
 $\downarrow \rightarrow a_n \in \mathcal{F} \Rightarrow A$ closed

(iii) A convex: let $\alpha, \beta \in A$, $t \in [0, 1]$,

$$y = t\alpha + (1-t)\cdot\beta.$$

 $(\frac{1}{2}, \frac{1}{2}, 0)$
 $(\frac{1}{2}, 0, \frac{1}{2})$

$$\begin{aligned} \text{Then } \min(y) &= \min(t\alpha + (1-t)\beta) \geq t \cdot \min(\alpha) + (1-t) \cdot \min(\beta) \\ &\geq t \cdot 0 + (1-t) \cdot 0 = 0. \end{aligned}$$

Similarly, $\max(y) \leq 1$. \Rightarrow entries in y shall $\in [0, 1]$

Let $\tilde{\alpha}, \tilde{\beta}, \tilde{y}$ be the parts of α, β, y that encode the probability distribution of player i .

$$\begin{aligned}
 \text{Then } \sum \tilde{\gamma} &= \sum (t \tilde{\alpha} + (1-t) \tilde{\beta}) \\
 &= t \cdot \underbrace{\sum \tilde{\alpha}}_{\text{m}} + (1-t) \underbrace{\sum \tilde{\beta}}_{\text{m}} \\
 &= t \cdot 1 + (1-t) 1 \\
 &= \cancel{1} 1
 \end{aligned}$$

\Rightarrow probabilities for player i in γ still add up to 1

$\Rightarrow \gamma \in \mathcal{A} \Rightarrow \mathcal{A}$ convex.

(iv) $B(\alpha) \neq \emptyset$ f. a. α :

for fixed α_{-i} , U_i is linear in strategies of player i : for $\beta_i, y_i \in \Delta(A_i)$:

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)y_i)$$

$$= \lambda \cdot U_i(\alpha_{-i}, \beta_i) + (1-\lambda) \cdot U_i(\alpha_{-i}, y_i)$$

for any $\lambda \in [0, 1]$



$\Rightarrow U_i$ continuous on $\Delta(A_i)$

$\Delta(A_i)$ compact

$\Rightarrow U_i$ has its maximum in $\Delta(A_i)$

$\Rightarrow B_i(\alpha_{-i}) \neq \emptyset \Rightarrow B(\alpha) \neq \emptyset$

(v) $B(\alpha)$ convex: let $\alpha_i^!, \alpha_i^{''} \in B_i(\alpha_{-i})$.

$$\Rightarrow U_i(\alpha_{-i}, \alpha_i^!) = U_i(\alpha_{-i}, \alpha_i^{''}) =: U_{\max}.$$

$$\Rightarrow U_i(\alpha_{-i}, \underline{\lambda \cdot \alpha_i^! + (1-\lambda) \cdot \alpha_i^{''}})$$

$$\stackrel{*}{=} \lambda \cdot \underline{U_i(\alpha_{-i}, \alpha_i^!)} + (1-\lambda) \cdot \underline{U_i(\alpha_{-i}, \alpha_i^{''})}$$

$$= \lambda \cdot U_{\max} + (1-\lambda) \cdot U_{\max} = U_{\max}$$

$$\Rightarrow \lambda \cdot \alpha_i^! + (1-\lambda) \cdot \alpha_i^{''} \in B_i(\alpha_{-i})$$

$\Rightarrow B_i(\alpha_{-i})$ convex $\Rightarrow B(\alpha)$ convex

(vi) Graph(B) closed: let (α^n, β^n) be a convergent sequence in Graph(B) with

$$\lim_{n \rightarrow \infty} (\alpha^n, \beta^n) = (\alpha, \beta).$$

i.e. $\alpha^n, \beta^n, \alpha, \beta \in \prod_{i \in N} \Delta(A_i)$ and $\beta^n \in B(\alpha^n)$. Need to show: $(\alpha, \beta) \in \text{Graph}(B)$, i.e. $\beta \in B(\alpha)$. For all players $i \in N$:

$$U_i(\alpha_{-i}, \beta_i) \stackrel{\text{Def.}}{=} U_i\left(\lim_{n \rightarrow \infty} (\alpha_{-i}^n, \beta_i^n)\right)$$

$$\begin{aligned} & \stackrel{\text{continuity}}{=} \lim_{n \rightarrow \infty} U_i(\alpha_{-i}^n, \beta_i^n) \\ & \stackrel{\text{best response}}{\geq} \lim_{n \rightarrow \infty} U_i(\alpha_{-i}^n, \beta_i^n) \quad \text{f.g. } \beta_i^n \\ & \quad \in \Delta(A_i) \end{aligned}$$

$$\dots \lim_{n \rightarrow \infty} U_i(\underline{\alpha}_{-i}, \beta_i') \quad \text{f.a. } \beta_i' \in \Delta(A_i)$$

continuity

$$= U_i(\underbrace{\lim_{n \rightarrow \infty} \alpha_{-i}^n}_{\alpha_{-i}}, \beta_i') \quad \text{f.a. } \beta_i' \in \Delta(A_i)$$

D.f. α

$$= U_i(\alpha_{-i}, \beta_i') \quad \text{f.a. } \beta_i' \in \Delta(A_i)$$

$$\Rightarrow \beta_i \in B(\alpha_{-i}) \Rightarrow \beta \in B(\alpha)$$

$$\Rightarrow (\alpha, \beta) \in \text{Graph}(B) \Rightarrow \text{Graph}(B) \text{ closed}$$

\Rightarrow Kakutani's approach

$\Rightarrow B$ has fixpoint $\alpha^* \in B(\alpha^*)$

$\Rightarrow \alpha^*$ is a MSNE

□