

### 3 Mixed Strategies

Def.: Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a finite strategic game. Let  $\Delta(A_i)$  be the set of all probability distributions over  $A_i$ . Then an  $\alpha_i \in \Delta(A_i)$  is a mixed strategy in  $G$ , where  $\underline{\alpha_i(a_i)}$  is the probability that player  $i$  chooses  $a_i \in A_i$ . A profile  $(\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$  induces a probability distribution over  $A = \prod_{i \in N} A_i$  as follows:

$$p(a) = \prod_{i \in N} \alpha_i(a_i).$$

Example : N P

		2	
		H	T
		H	1, -1
1		-1, 1	-1, 1
→		-1, 1	1, -1

For player 1:  $\alpha_1(H) = \frac{2}{3}, \alpha_1(T) = \frac{1}{3}$

For player 2:  $\alpha_2(H) = \frac{1}{3}, \alpha_2(T) = \frac{2}{3}$ .

$$\text{Then : } P(H, H) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} \quad u_1(H, H) = +1$$

$$P(H, T) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} \quad u_1(H, T) = -1$$

$$P(T, H) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \quad u_1(T, H) = -1$$

$$P(T, T) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \quad u_1(T, T) = +1$$

$$u_1, \text{ expected : } \frac{2}{9} \cdot (+1) + \frac{4}{9} \cdot (-1) + \frac{1}{9} \cdot (-1) + \frac{2}{9} \cdot (+1)$$
$$= -\frac{1}{9}$$

$$u_2, \text{ expected : } +\frac{1}{9}$$

Def.: Let  $\alpha \in \prod_{i \in N} \Delta(A_i)$ . Then the expected utility (expected payoff) of  $\alpha$  for player  $i$  is

$$U_i(\alpha) = \sum_{a \in A} \underbrace{\left( \prod_{i \in N} \alpha_i(a_i) \right)}_{p(a)} \cdot u_i(a)$$

Example: In previous example,  $U_1(\alpha_1, \alpha_2) = -\frac{1}{9}$ ,

$$U_2(\alpha_1, \alpha_2) = +\frac{1}{9}$$

Def., The mixed extension of a (f Nash) strategy game  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  is the game  $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$  where  $\Delta(A_i)$  are the prob. dist. over  $A_i$  and  $U_i$  exp. utility function of player  $i$ .

Def.: Let  $\alpha_i$  be a mixed strategy. The support of  $\alpha_i$  is the set  $\text{Supp}(\alpha_i) = \{a_i \in A_i \mid \alpha_i(a_i) > 0\}$ .

Def.: Let  $G$  be a strategic game. A Nash equilibrium in mixed strategies (mixed strategy NE, MSNE) is a Nash equilibrium in the mixed extension of  $G$ .

Lemma: Let  $G = \langle N, (A_i)_{i \in N}, (\alpha_i)_{i \in N} \rangle$  be a (finite) strategic game. Then  $\alpha^* \in \prod_{i \in N} \Delta(A_i)$  is a MSNE of  $G$  iff for each player  $i \in N$  every pure strategy in  $\text{supp}(\alpha_i^*)$  is a best response to  $\alpha_{-i}^*$ .

Example: tip,  $\alpha_1(H) = \frac{2}{3}$ ,  $\alpha_1(T) = \frac{1}{3}$

$$\underline{\alpha_2(H) = \frac{1}{3}, \alpha_2(T) = \frac{2}{3}}$$

NSNE? No!

$$\begin{aligned} u_2(\alpha_1, H) &= \frac{2}{3} \cdot u_2(H, H) + \frac{1}{3} \cdot u_2(T, H) \\ &= \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot (+1) = -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} u_2(\alpha_1, T) &= \frac{2}{3} \cdot u_2(H, T) + \frac{1}{3} \cdot u_2(T, T) \\ &= \frac{2}{3} \cdot (+1) + \frac{1}{3} \cdot (-1) = +\frac{1}{3} \end{aligned}$$

$$\begin{aligned} B_2(\alpha_1) &= \{T\}, \quad \left| \begin{array}{l} H \notin B_2(\alpha_1), \\ H \in \text{supp}(\alpha_2) \end{array} \right\} \Rightarrow (\alpha_1, \alpha_2) \\ \text{but} \quad & \end{aligned}$$

not NSNE.

Proof (Lemma): Let  $\alpha^*$  be a MNE and  
 $a_i \in \text{supp}(\alpha_i^*)$ .

" $\Rightarrow$ ":

Suppose  $a_i$  is not a best response  
to  $\alpha_{-i}^*$ . Then ex.  $a'_i \in A_i$  s.t.

$$U_i(\alpha_{-i}^*, a'_i) > U_i(\alpha_{-i}^*, a_i).$$

Then  $a'_i$  with weight shifted from  $a_i$   
to  $a'_i$  would be a better response to  
 $\alpha_{-i}^*$  than  $\alpha_i^*$  is. Then  $\alpha_i^* \notin B_i(\alpha_{-i}^*)$ .

By assumption that  $\alpha^*$  is MNE.

" $\leq$ ", Suppose that  $\alpha^*$  is not a MNE.

Then there exists player  $i$  and a mixed strategy  $\alpha_i'$  such that

$$U_i(\underline{\alpha_{-i}^*}, \underline{\alpha_i'}) > U_i(\underline{\alpha_{-i}^*}, \underline{\alpha_i^*}).$$

Then (because of the linearity of  $U_i$ ) there exists a strategy  $a_i' \in A_i$  in  $\text{supp}(\alpha_i')$  with higher payoff against  $\alpha_{-i}^*$  than at least one pure strategy  $a_i \in \text{supp}(\alpha_i^*)$ .  
I.e. not all  $a_i \in \text{supp}(\alpha_i^*)$  are best responses to  $\alpha_{-i}^*$ .

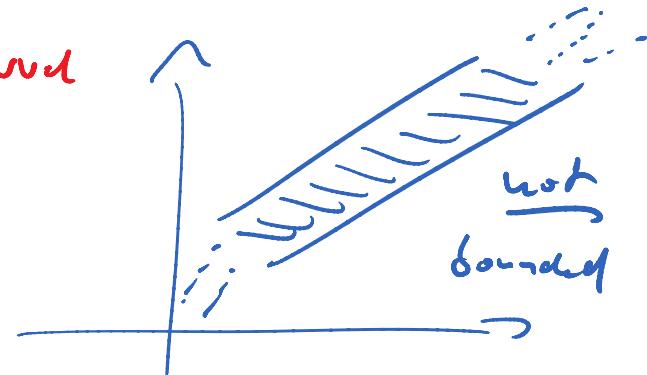
## Existence of MSNE

Theorem (Nash): Every finite strategic game has an MSNE.

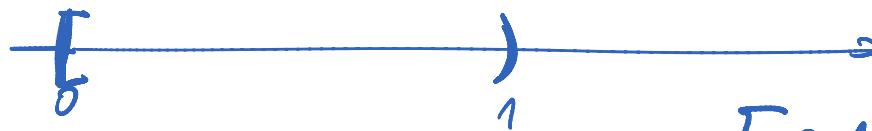
Proof: later

Def.: (a)  $X \subseteq \mathbb{R}^n$  is bounded if for each  $1 \leq i \leq n$  ex.  $a_i, b_i \in \mathbb{R}$  such that

$$X \subseteq \prod_{i=1}^n [a_i, b_i].$$



(b)  $X \subseteq \mathbb{R}^n$  is closed if the limit of each convergent sequence of elements of  $X$  is contained in  $X$ .

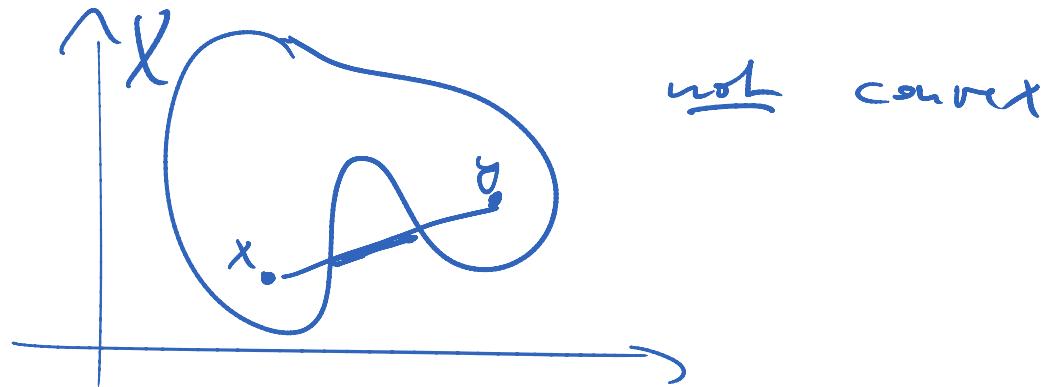


$[0, 1)$  is not closed

Reason:  $1, 1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, \dots \rightarrow 1 \notin [0, 1)$

(c)  $X \subseteq \mathbb{R}^n$  is compact if it is bounded and closed.

(d)  $X \subseteq \mathbb{R}^n$  is convex if for each  $x, y \in X$ , and for any  $t \in [0, 1]$ ,  
also  $(1-t)x + t \cdot y \in X$



(e) For function  $f: X \rightarrow 2^X$ , the graph of  $f$  is the set

$$\text{Graph}(f) = \{(x, y) \mid x \in X, y \in f(x)\}.$$

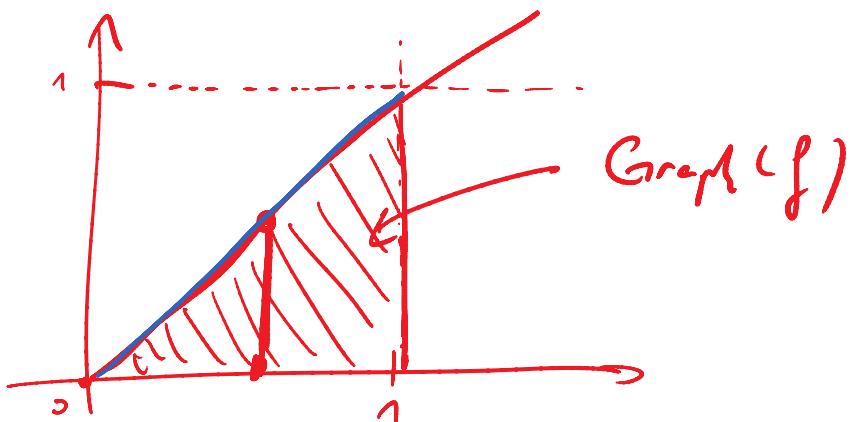
Theorem (Kakutani): Let  $X \subseteq \mathbb{R}^n$  be a nonempty, compact and convex set and  $f: X \rightarrow 2^X$  be a function such that

- (i) for each  $x \in X$ , the set  $f(x) \subseteq X$  is nonempty and convex, and
- (ii)  $\text{Graph}(f)$  is closed.

Then ex.  $x \in X$  with  $x \in f(x)$ , i.e.,  
 $f$  has a fixpoint. □

Prof: e.g. Hennig, Lehrbuch der Realanalysis, Bd. 2 . 1

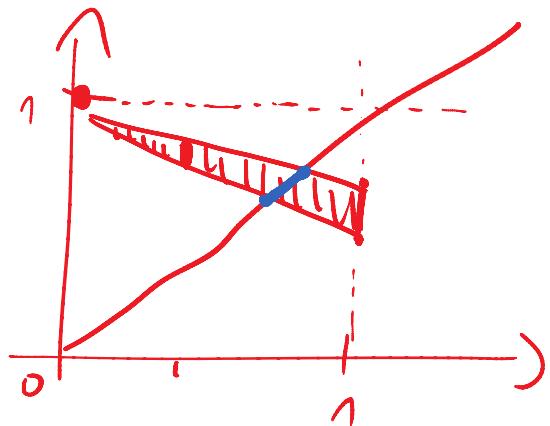
Example: (a)  $f : [0,1] \rightarrow 2^{[0,1]}$ ,  $f(x) = \{y \mid y \leq x\}$



$$x \in \{y \mid y \leq x\}$$

since  $x \subseteq x$

(b)  $f : [0,1] \rightarrow 2^{[0,1]}$ ,  $f(x) = \{y \mid 1 - \frac{x}{2} \leq y \leq 1 - \frac{x}{4}\}$



Proof (Mark): Encode (profiles of) mixed strategies as real-valued vectors; e.g.

$$\alpha = (\alpha_1, \alpha_2) \text{ with } \alpha_1(T) = 0.3, \alpha_1(M) = 0.2,$$

$$\alpha_1(B) = 0.5, \alpha_2(L) = 0.6, \alpha_2(R) = 0.4$$

is encoded as  $(0.3, 0.2, 0.5, 0.6, 0.4)$

$\in [0,1]^5 \subseteq \mathbb{R}^5$ . Encode best-response functions as  $B: \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ ,

$B(\alpha) = \bigcap_{i \in N} B_i(\alpha_{-i})$ . Then  $\alpha$  is fixpoint of  $B$  iff  $\alpha \in B(\alpha)$  iff  $\alpha$  is MSNE.

We have to show that Kakutani's fixpoint theorem is applicable for  $X = \overline{\bigcap_{i \in N} A_i}$   
and  $f = \mathcal{B}$ .