

### 3 Mixed Strategies

Def.: Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a finite strategy game. Let  $\Delta(A_i)$  be the set of all probability distributions over  $A_i$ . Then an  $\alpha_i \in \Delta(A_i)$  is a mixed strategy in  $G$ , where  $\alpha_i(a_i)$  is the probability that player  $i$  chooses  $a_i \in A_i$ . A profile  $(\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$  induces a probability distribution over  $A = \prod_{i \in N} A_i$  as follows:

$$\text{Then: } p(H, H) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} \quad u_1(H, H) = +1$$

$$p(H, T) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} \quad u_1(H, T) = -1$$

$$p(T, H) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \quad u_1(T, H) = -1$$

$$p(T, T) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \quad u_1(T, T) = +1$$

$$u_1, \text{expected: } \frac{2}{9} \cdot (+1) + \frac{4}{9} \cdot (-1) + \frac{1}{9} \cdot (-1) + \frac{2}{9} \cdot (+1) \\ = -\frac{1}{9}$$

$$u_2, \text{expected: } +\frac{1}{9}$$

$$p(a) = \prod_{i \in N} \alpha_i(a_i).$$

Example:  $\rightarrow$

		2
	H	T
1	1, -1	-1, 1

		2
	H	T
1	1, -1	-1, 1

		2
	H	T
1	1, -1	-1, 1

$$\text{For player 1: } \alpha_1(H) = \frac{2}{3}, \quad \alpha_1(T) = \frac{1}{3}$$

$$\text{For player 2: } \alpha_2(H) = \frac{1}{3}, \quad \alpha_2(T) = \frac{2}{3}.$$

Def.: Let  $\alpha \in \prod_{i \in N} \Delta(A_i)$ . Then the expected utility (expected payoff) of  $\alpha$  for player  $i$  is

$$U_i(\alpha) = \sum_{a \in A} \underbrace{\left( \prod_{i \in N} \alpha_i(a_i) \right)}_{p(a)} \cdot u_i(a)$$

$$\text{Example: In previous example, } U_1(\alpha_1, \alpha_2) = -\frac{1}{9},$$

$$U_2(\alpha_1, \alpha_2) = +\frac{1}{9}.$$

Def.: The mixed extension of a (finsh) strategic game  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  is the game  $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$  where  $\Delta(A_i)$  are the prob. dist. over  $A_i$  and  $U_i$  exp. utility function of player  $i$ .

Def.: Let  $\alpha_i$  be a mixed strategy. The support of  $\alpha_i$  is the set  $\text{supp}(\alpha_i) = \{a_i \in A_i \mid \alpha_i(a_i) > 0\}$ .

Example:  $\alpha_1(H) = \frac{2}{3}, \alpha_1(T) = \frac{1}{3}$   
 $\underline{\alpha_2(H) = \frac{1}{3}, \alpha_2(T) = \frac{2}{3}}$

MNE? No!

$$\begin{aligned} u_2(\alpha_1, H) &= \frac{2}{3} \cdot u_2(H, H) + \frac{1}{3} \cdot u_2(T, H) \\ &= \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot (-1) = -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} u_2(\alpha_1, T) &= \frac{2}{3} \cdot u_2(H, T) + \frac{1}{3} \cdot u_2(T, T) \\ &= \frac{2}{3} \cdot (+1) + \frac{1}{3} \cdot (-1) = +\frac{1}{3} \end{aligned}$$

$$B_2(\alpha_1) = \{T\}, \quad \left| \begin{array}{l} H \notin B_2(\alpha_1), \\ H \in \text{supp}(\alpha_2) \end{array} \right\} \Rightarrow (\alpha_1, \alpha_2) \text{ not MNE.}$$

Def.: Let  $G$  be a strategic game. A Nash equilibrium in mixed strategies (mixed strategy NE, MSNE) is a Nash equilibrium in the mixed extension of  $G$ .

Lemma: Let  $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a (finsh) strategic game. Then  $\alpha^* \in \prod_{i \in N} \Delta(A_i)$  is a MSNE of  $G$  iff for each player  $i \in N$  every pure strategy in  $\text{supp}(\alpha_i^*)$  is a best response to  $\alpha_{-i}^*$ .

Proof (Lemma): Let  $\alpha^*$  be a MSNE and  $a_i \in \text{supp}(\alpha_i^*)$ .  
 $\Rightarrow$ : Suppose  $a_i$  is not a best response to  $\alpha_{-i}^*$ . Then ex.  $a'_i \in A_i$  s.t.  $u_i(\alpha_{-i}^*, a'_i) > u_i(\alpha_{-i}^*, a_i)$ . Then  $\alpha'_i$  with weight shifted from  $a_i$  to  $a'_i$  would be a better response to  $\alpha_{-i}^*$  than  $\alpha_i^*$  is. Then  $\alpha_i^* \notin B_i(\alpha_{-i}^*)$ . By assumption that  $\alpha^*$  is MNE.

$\Leftarrow$ : Suppose that  $\alpha^*$  is not a MSNE.

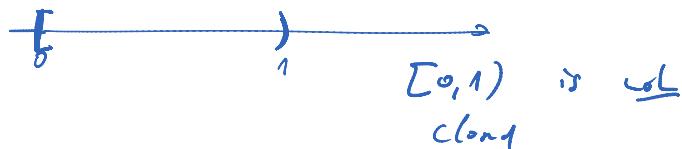
Then there exists player  $i$  and a mixed strategy  $\alpha'_i$  such that

$$U_i(\alpha_{-i}^*, \underline{\alpha'_i}) > U_i(\alpha_{-i}^*, \underline{\alpha_i^*}).$$

Then (because of the linearity of  $U_i$ ) there exists a strategy  $a'_i \in A_i$  in  $\text{supp}(\alpha'_i)$  with higher payoff against  $\alpha_{-i}^*$  than at best one pure strategy  $a_i \in \text{supp}(\alpha_i^*)$ .

I.e. not all  $a_i \in \text{supp}(\alpha_i^*)$  are best responders to  $\alpha_{-i}^*$ .

(b)  $X \subseteq \mathbb{R}^n$  is closed if the limit of each convergent sequence of elements of  $X$  is contained in  $X$ .



Reason:  $1, 1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, \dots \rightarrow 1 \notin [0,1]$

(c)  $X \subseteq \mathbb{R}^n$  is compact if it is bounded and closed.

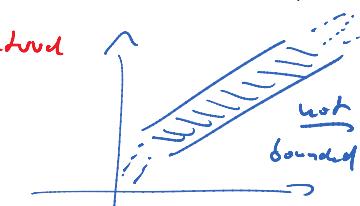
### Existence of MSNE

Theorem (Nash): Every finite strategic game has a MSNE.

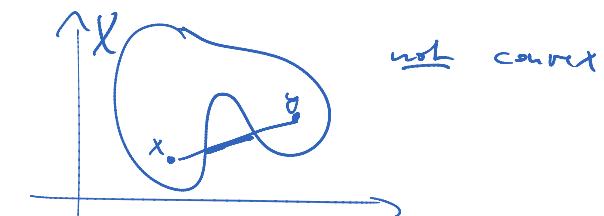
Proof: later

Def.: (a)  $X \subseteq \mathbb{R}^n$  is bounded if for each  $1 \leq i \leq n$  ex.  $a_{ii}, b_i \in \mathbb{R}$  such that

$$X \subseteq \prod_{i=1}^n [a_{ii}, b_i].$$



(d)  $X \subseteq \mathbb{R}^n$  is convex if for each  $x, y \in X$ , and for any  $t \in [0,1]$ , also  $(1-t)x + t \cdot y \in X$



(e) For function  $f: X \rightarrow 2^X$ , the graph of  $f$  is the set

$$\text{Graph}(f) = \{(x, y) \mid x \in X, y \in f(x)\}.$$

Theorem (Kakutani): Let  $X \subseteq \mathbb{R}^n$  be a nonempty, compact and convex set and  $f: X \rightarrow 2^X$  be a function such that

(i) for each  $x \in X$ , the set  $f(x) \subseteq X$

is nonempty and convex, and

(ii)  $\text{Graph}(f)$  is closed.

Then ex.  $x \in X$  with  $x \in f(x)$ , i.e.,

$f$  has a fixpoint.  $\square$

Prof: e.g. Hensel, Lehrbuch der Dyn. Sys., Bd. 2.  $\square$

Proof (Nat): Encode (profiles of) mixed strategies as real-valued vectors; e.g.

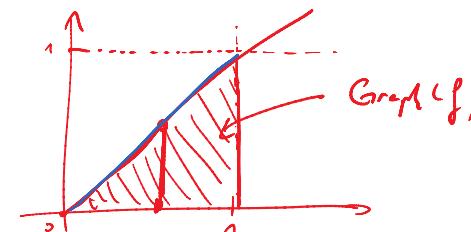
$\alpha = (\alpha_1, \alpha_2)$  with  $\alpha_1(T) = 0.3, \alpha_1(M) = 0.2,$   
 $\alpha_1(B) = 0.5, \alpha_2(L) = 0.6, \alpha_2(R) = 0.4$

is encoded as  $(0.3, 0.2, 0.5, 0.6, 0.4)$

$\in [0,1]^5 \subseteq \mathbb{R}^5$ . Encode best-response functions as  $B: \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$ ,

$B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i})$ . Then  $\alpha$  is fixpoint of  $B$  iff  $\alpha \in B(\alpha)$  iff  $\alpha$  is MSNE.

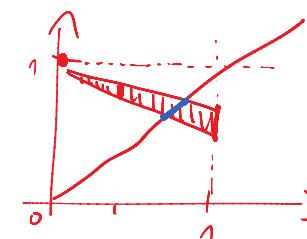
Example: a)  $f: [0,1] \rightarrow 2^{[0,1]}$ ,  $f(x) = \{y \mid y \leq x\}$



$$x \in \{y \mid y \leq x\}$$

since  $x \leq x$

b)  $f: [0,1] \rightarrow 2^{[0,1]}$ ,  $f(x) = \{y \mid 1 - \frac{x}{2} \leq y \leq 1 - \frac{x}{4}\}$



We have to show that Kakutani's fixpoint theorem is applicable for  $X = \prod_{i \in N} \Delta(A_i)$  and  $f = B$ .