Lemma: Let G be a ZSG. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x,y). \quad \square$$

Maximin/minimax Theorem:

(a) Whenever (x^*, y^*) is a NE of a ZSG G, x^* and y^* are MM of player 1 and 2, respectively.

(b) If (x^*, y^*) is a NE of ZSG G, then

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) = u_1(x^*, y^*).$$

This means all NE of a ZSG have same payoff.

(c) If $\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$ and x^* and y^* are MM for player 1 and 2, then (x^*, y^*) is NE of ZSG G.

Corollary: If (x_1^*, y_1^*) and (x_2^*, y_2^*) are NE of ZSG G, then (x_1^*, y_2^*) and (x_2^*, y_1^*) are NE of G, too.

Proof (Thm.):

(a) Let (x^*, y^*) be NE.

(i) NE $\Rightarrow u_2(x^*, y^*) \geq u_2(x^*, y)$ f.a. $y \in A_2$

$$\overset{u_1 = -u_2}{\Longrightarrow} \quad u_1(x^*, y^*) \leq u_1(x^*, y) \text{ f.a. } y \in A_2$$

$$\Rightarrow \quad u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y)$$

$$\leq \max_{x \in A_1} \min_{y \in A_2} u_1(x,y) \quad (+)$$

(ii) NE $\Rightarrow u_1(x^*, y^*) \geq u_1(x, y^*)$ f.a. $x \in A_1$

$$\Rightarrow u_1(x^*, y^*) \geq \min_{y \in A_2} u_1(x,y) \text{ f.a. } x \in A_1$$

$$\Rightarrow u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x,y). \quad (++)$$

$$(+) \wedge (++) \Rightarrow u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x,y).$$

$\Rightarrow x^*$ is MM for player 1.

Analogously: $u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$.

$\Rightarrow y^*$ is MM for player 2.

(b) From (a): $u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x,y)$

$$\parallel$$

and $u_1(x^*, y^*) = -u_2(x^*, y^*) = -\max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$

$$= \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$$

(c) Let x^* and y^* be MM for players 1/2,

and $\max_{x \in A_1} \min_{y \in A} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*$.

Lemma $\Rightarrow$ $-v^* = \max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$.

x^*, y^* MM $\Rightarrow$ $u_1(x^*, y) \geq v^*$ f.a. $y \in A_2$ (*)

and $u_2(x, y^*) \geq -v^*$ f.a. $x \in A_1$. (**)

With $y = y^*$ and $x = x^*$:

$u_1(x^*, y^*) \geq v^*$

$u_2(x^*, y^*) \geq -v^* \overset{u_1 = -u_2}{\Rightarrow} u_1(x^*, y^*) \leq v^*$

$\Rightarrow u_1(x^*, y^*) = v^*$ (***)

(*) $\wedge$ (***) $\Rightarrow u_1(x^*, y) \geq u_1(x^*, y^*)$ f.a. $y \in A_2$.

$\overset{u_2 = -u_1}{\Rightarrow} u_2(x^*, y) \leq u_2(x^*, y^*)$ f.a. $y \in A_2$.

$\Rightarrow y^* \in B_2(x^*)$.

(**) $\wedge$ (***) $\Rightarrow u_2(x, y^*) \geq u_2(x^*, y^*)$ f.a. $x \in A_1$

$\overset{u_1 = -u_2}{\Rightarrow} u_1(x, y^*) \leq u_1(x^*, y^*)$ f.a. $x \in A_1$

$\Rightarrow x^* \in B_1(y^*)$.

$\Rightarrow (x^*, y^*)$ NE. $\square$

<u>Proof (Corollary)</u>: Let (x_1^*, y_1^*) and (x_2^*, y_2^*) are NE.

(i) With (a): x_1^*, x_2^* MM for 1, y_1^*, y_2^* MM for 2.

(ii) With (b): $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$.

With (i) $\wedge$ (ii) $\wedge$ (c): (x_1^*, y_2^*) and (x_2^*, y_1^*) NE. $\square$