Sealed-Bid Auctions

An object has to be assigned to one player $i \in \{1, \dots, n\}$ in exchange for a payment. For each player i, v_i is the valuation of player i of the object. W.l.g. we will assume that

$$v_1 > v_2 > v_3 > \dots > v_n .$$

<u>Mechanisms</u>: Player give simultaneously their bids $b_1, b_2, \dots, b_n \geq 0$. The object is assigned to player with the highest bid. Break ties by valuation order, i.e., if $b_i = b_j$ are the highest bids, then i will win iff $i < j$.

First price auction: The payment by the winner is his bid.

Second price auction: The payment by the winner is the highest bid of the non-winning agents.

Formulization of these games:

$N = \{1, \dots, n\}$

$A_i = \{ b_i \mid b_i \in \mathbb{R}^+ \}$

$$u_i(b) = \begin{cases} 0 & \text{, if the player } i \text{ does not win} \\ v_i - b_i & \text{, otherwise} \end{cases}$$

for first price auction.

For second price auction:

N, A_i for each i is the same

$$u_i(b) = \begin{cases} 0 & \text{, if } i \text{ does not win} \\ v_i - \max b_{-i} & \text{, otherwise} \end{cases}$$

<u>Example</u>: Three players: 1, 2, 3.

$v_1 = 100$, $v_2 = 80$, $v_3 = 55$

$b_1 = 90$ $b_2 = 85$, $v_3 = 45$

1 wins yields the utility:

$u_1(b) = v_1 - b_1 = 10$, if first price auction

$u_1(b) = v_1 - \max b_{-1} = 15$, if second price auction

Proposition In a second price auction, bidding your own valuation, b_i^+, is a weakly dominating strategy.

Proof:

1) Regardless of what the others agent do, b_i^+ is always the best strategy:

i wins: i has to pay $\max b_{-i} \leq v_i$, which means that $u_i(b_{-i}, b_i^+) \geq 0$. Lowering our bid cannot improve the payoff, but might do loosing the auction. Increasing our bid does not help either.

i looses: $u_i(b_{-i}, b_i^+) = 0$. Lowering b_i^+ does not change anything. By increasing his bid, he can win, he will have to pay a price $\geq b_i^+ = v_i$, i.e. either 0 or negative utility.

2) b_i^+ is strictly better than any othe strategy under some profile.

Let b_i' some strategy $\neq b_i^+$.

$b_i' < b_i^+$: Now let us consider b_{-i} with $\max b_{-i} > b_i'$, $b_i^+ >$
with b_i' we do not win, i.e., we have $u_i(b_{-i}, b_i') = 0$,
while with b_i^+: $u_i(b_{-i}, b_i^+) > 0$.

$b_i' > b_i^+$: Consider $b_i' > \max b_{-i} > b_i^+$. Here
$u_i(b_{-i}, b_i') < 0$ and $u_i(b_{-i}, b_i^+) = 0$.

<u>Remark</u>: A profile of weakly dominating strategie is a NE, because for nobody there is an icentive to deviate.

<u>Remark</u>: There is a second NE for second price auctions! This is $b = (v_1, v_1, \dots, v_1)$. For

- v_1: If he lowers, he does not win, so utility is still $\emptyset$. If he increases, he still wins and has to pay v_1, utility is still $\emptyset$.
- For all others: Increasing leads to negative utility. Decreasing does not change anything, since they do not win.

2.6 Zero Sum Games and NE

<u>Def</u> (Zero sum game)

A zero sum game (ZSG) is a 2 player strategic game

$$G = \langle \{1,2\}, (A_i)_{i \in \{1,2\}}, (u_i)_{i \in \{1,2\}} \rangle$$

such that for all profiles $a \in A$: $u_1(a) + u_2(a) = 0$.

<u>Remark</u>: Can be generalized to constant sum games, where the utilities sum up to some constant c.

	L	M	R	
T	8,-8	3,-3	-6,6	→ -6
M	2,-2	-1,1	3,-3	→ -1
B	-6,6	4,-4	8,-8	→ -6
	-8	-4	-8	

<u>Idea</u>: Try to play it safe.

Let us assume, the other player does as much harm as he can. Then maximize over the outcomes.

<u>Def</u> Let G be ZSG. $x^* \in A_1$ is called a <u>maximiniminizer</u> for player 1, if:

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \text{ for all } x \in A_1$$

Similar for player 2.

Example

	L	R	
T	1, -1	2, -2	→ 1
B	-2, 2	-4, 4	→ -4
	↓	↓	
	-1	-2	

Profile (T, L) is a NE and it is a pair of maximinizers.

We will show that each NE in ZSG is a pair of maximinimizers.

<u>Lemma</u> Let G be a ZSG. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$$

<u>Proof</u>: For each real valued function f, it holds

(1) $$\min_z (-f(z)) = -\max_z (f(z))$$

Thus:

$$(2) \quad -\min_{x \in A_1} u_2(x,y) \overset{(1)}{=} \max_{x \in A_1} -u_2(x,y)$$

$$\overset{u_1 = -u_2}{=} \max_{x \in A_1} u_1(x,y)$$

Thus:

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) \overset{(1)}{=} -\min_{y \in A_2} -\left(\min_{x \in A_1} u_2(x,y)\right)$$

$$= -\min_{y \in A_2} \max_{x \in A_1} u_1(x,y) \qquad \square$$

Maximinimizer Theorem

(a) Whenever (x^*, y^*) is a NE of a ZSG G, then x^* and y^* are maximinimizers of player 1 and player 2, respectively.

(b) If (x^*, y^*) is a NE of a ZSG G, then

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$$
$$= u_1(x^*, y_2^*)$$

This means all NE in ZSG have the same payoff.

(c) if $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$ and x^* and y^* are maximinimizers for player 1 and player 2, respectively, then (x^*, y^*) is a NE.

In particular if (x_1^*, y_1^*) and (x_2^*, y_2^*) are NE, then so is (x_1^*, y_2^*) and (x_2^*, y_1^*).

~~Proof~~: