Introduction to Game Theory

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2.8 Mixed Strategies

In this section we consider a class of infinite strategic games that are induced by finite strategic games, when the players' choices are determined by some probabilistic rules.

Notation: A probability distribution over a finite set $X \neq \emptyset$ is any function

$$p : X \to [0, 1]$$

such that $\sum_{x \in X} p(x) = 1$. The set of probability distributions over X is denoted by ΔX.

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game. The idea of so-called "mixed strategies" is as follows:

* a strategic game is played iteratively
* the players initially choose a probabilistic rule according to which each round is played.

A mixed strategy, then, is a choice of such a probabilistic rule. More formally, a <u>mixed strategy of player i</u> in game G is a probability distribution over A_i. ΔA_i denotes the set of all mixed strategies.

<u>Note</u>: Even though G is finite, i.e. all sets A_i are finite, the sets ΔA_i are infinite.

The strategies in A_i are referred to as <u>pure strategies</u>.

Moreover, each $a_i \in A_i$ induces a particular mixed strategy, namely

$$\hat{a}_i(a_i') := \begin{cases} 1 & \text{if } a_i = a_i' \\ 0 & \text{else} \end{cases}$$

Each profile of mixed strategies $\overset{=\alpha}{(\alpha_i)_{i\in N}} \in \prod_{i\in N} \Delta A_i$

induces a probability distribution on $A = \prod_{i\in N} A_i$

$$p(a) = \prod_{i\in N} \alpha_i(a_i) = \alpha_1(a_1)\cdot\ldots\cdot\alpha_{|N|}(a_{|N|})$$

– the probability that profile a is played.

For any set $A' \subseteq A$, define

$$p(A') := \sum_{a\in A'} p(a) = \sum_{a\in A'} \left(\prod_{i\in N} \alpha_i(a_i)\right).$$

<u>Definition</u>: Let G be finite.

Let $\alpha \in \prod_{i\in N} \Delta A_i$ be a profile of mixed strategies.

The <u>expected utility of α</u> for player i in G is defined as:

$$U_i(\alpha) = U_i((\alpha_j)_{j\in N})$$

$$:= \sum_{a\in A} p(a)\cdot u_i(a)$$

$$= \sum_{a\in A} \left(\prod_{j\in N} \alpha_j(a_j)\right)\cdot u_i(a).$$

$U_i : \prod_{i\in N} \Delta A_i \longrightarrow \mathbb{R}$

Example: Mixed strategies in Matching Pennies

$H \to$ head, $T \to$ tail

	head	tail
H	1, −1	−1, 1
T	−1, 1	1, −1

Player 1 chooses $\alpha_1 \in \Delta\{H,T\}$ with

$$\alpha_1(H) = 2/3, \quad \alpha_1(T) = 1/3$$

Player 2 chooses $\alpha_2 \in \Delta\{H,T\}$ with

$$\alpha_2(H) = 1/3, \quad \alpha_2(T) = 2/3.$$

Shorthand for specifying mixed strategy profiles

$$\left(2/3 \cdot H + 1/3 \cdot T,\ 1/3 \cdot H + 2/3 \cdot T\right)$$

Thus, we have the following prob. distributions on A:

$$p(H,H) = \alpha_1(H)\cdot\alpha_2(H) = 2/9$$
$$p(H,T) = \ldots = 4/9$$
$$p(T,H) = \ldots = 1/9$$
$$p(T,T) = \ldots = 2/9$$

Thus,

$$u_1(\alpha_1,\alpha_2) = \sum_{a\in A} p(a)\cdot u_i(a)$$
$$= 2/9 \cdot 1 + 4/9\cdot(-1) + 1/9\cdot(-1) + 2/9\cdot 1$$
$$= -1/9$$

$$u_2(\alpha_1,\alpha_2) = \ldots = 1/9$$

Remark: The expected utility functions u_i are multi-linear, given $\alpha \in \prod_i \Delta A_i$, $\beta_i, \gamma_i \in \Delta A_i$, $\lambda \in [0,1]$, it follows

$$u_i(\alpha_{-i}, \lambda \beta_i + (1-\lambda)\gamma_i) =$$

$$\lambda \cdot u_i(\alpha_{-i}, \beta_i) + (1-\lambda)\, u_i(\alpha_{-i}, \gamma_i)$$

Prove it! Easy!

Moreover,

$$u_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot u_i(\alpha_{-i}, \hat{a}_i)$$

Prove it!

Example: In the Matching Pennies example with $\alpha = (\alpha_1, \alpha_2) = (2/3 \cdot H + 1/3 \cdot T,\ 1/3 \cdot H + 2/3 \cdot H)$ we can calculate $u_1(\alpha_1, \alpha_2)$ as follows:

$$u_1(\alpha_1, \alpha_2) = \alpha_1(H) \cdot u_1(\hat{H}, \alpha_2) + \alpha_1(T) \cdot u_1(\hat{T}, \alpha_2)$$

$$= 2/3 \cdot (1/3 - 2/3) + 1/3 \cdot (-1/3 + 2/3)$$

$$= \ldots = -1/9$$

Definition (Mixed extension):

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game. The mixed extension of G is the strategic game

$$G^{\Delta} = \langle N, (\Delta A_i)_{i \in N}, (U_i)_{i \in N} \rangle$$

where for each $i \in N$, ΔA_i is the set of all mixed strategies of player i in G, and $U_i : \prod_{j \in N} \Delta A_j \to \mathbb{R}$ is the function that assigns to each mixed strategy profile α the expected utility $U_i(\alpha)$ of α for player i.

Definition (Mixed strategy NE)

A mixed strategy NE of a strategic game G is a NE of the mixed extension of G.

Lemma: Each NE $a^* = (a_1^*, \ldots, a_{|N|}^*)$ of a finite strategic game G defines a NE

$$\hat{a}^* = (\hat{a}_1^*, \ldots, \hat{a}_{|N|}^*) \in \prod_{j \in N} \Delta A_j$$

of G^Δ.

Conversely, given $(a_1, \ldots, a_{|N|}) \in A$ such that

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_{|N|})$$

is a NE of G^Δ, then $(a_1, \ldots, a_{|N|})$ is a NE of G.

Proof sketch: Easy! Just observe that for all

$a = (a_1, \ldots, a_{|N|}) \in A$ it follows

$$U_i(\hat{a}) \underset{\text{of } U_i}{\overset{\text{def}}{=}} \sum_{b \in A} \left(\prod_{j \in N} \hat{a}_j(b_j) \right) \cdot u_i(b)$$

$$= u_i(a)$$

Then go on!

...

Let α_i be a mixed strategy of player i in the finite game G.

The support of α_i is defined as:

$$\text{support}(\alpha_i) := \{a_i \in A_i : \alpha_i(a_i) > 0\}$$

Lemma (Support lemma):

A mixed strategy profile $\alpha^* \in \prod_{i \in N} \Delta A_i$ is a mixed strategy NE if and only if for each player $i \in N$, each pure strategy $a_i \in \text{support}(\alpha_i^*)$ is a best response to α_{-i}^*, i.e.,

$$\hat{a}_i \in B_i(\alpha_{-i}^*)$$

Proof: Do it yourself!

Example: Consider Bach or Stravinsky (BoS)

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

2 NE in pure strategies: (B, B), (S, S).

Thus $(\hat{B}, \hat{B})$ and $(\hat{S}, \hat{S})$ are mixed strategy NE of the game.

Assume that (α_1, α_2) is any mixed strategy NE.

It follows

$$U_1(\hat{B}, \alpha_2) = \alpha_2(B) \cdot 2 + \alpha_2(S) \cdot 0 = 2\alpha_2(B)$$

Support-Lemma $\|$

$$U_1(\hat{S}, \alpha_2) = \alpha_2(B) \cdot 0 + \alpha_2(S) \cdot 1 = \alpha_2(S) = 1 - \alpha_2(B)$$

Thus, $\alpha_2(B) = 1/3$

In a similar way

$$u_2(\alpha_1, \hat{B}) = \dots = \alpha_1(B)$$

Support Lemma

$$u_2(\alpha_1, \hat{S}) = \dots = 2 \cdot \alpha_1(S) = 2 \cdot (1 - \alpha_1(B))$$

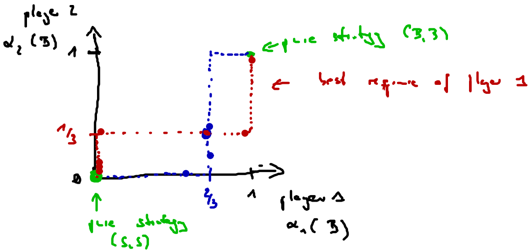
Thus, $\alpha_1(B) = 2/3$.

Thus,

$$\left(2/3 \cdot B + 1/3 \cdot S,\ 1/3 \cdot B + 2/3 \cdot S\right),$$

can be the only further mixed strategy NE of BoS.

Pictorial representation of BoS:



$\Rightarrow \left(\frac{2}{3}B + \frac{1}{3}S, \frac{1}{3}B + \frac{2}{3}S\right)$ is a NE

$$u_1(\alpha_1,\alpha_2) = \alpha_1(B)\cdot\alpha_2(B)\cdot 2 + \alpha_1(B)\cdot\alpha_2(S)\cdot 0 + \alpha_1(S)\cdot\alpha_2(B)\cdot 0 + \overbrace{\alpha_1(S)}^{1-\alpha_1(B)}\cdot\alpha_2(S)\cdot 1$$

$$= \dots$$

$$= 3\cdot\overbrace{\alpha_1(B)}^{x}\cdot\alpha_2(B) + 1 - \alpha_1(B) - \underbrace{\alpha_2(B)}_{:=b}$$

So we need to find maxima of the function:

$$f_b(x) := (3b-1)\cdot x + (1-b) \qquad b\in[0,1] \qquad x\in[0,1]$$

For $b<1/3$, $f_b(x)$ has max at $x=0$, because $3b-1<0$

For $b=1/3$, $f_b(x)$ has max constant for all $x\in[0,1]$

For $b>1/3$ $f_b(x)$ has max at $x=1$

Theorem : (Nash)

Each finite strategic game has a mixed strategy NE.

Proof: Let $G = \langle N, (A_i), (u_i) \rangle$ be a finite strategic game. Define $m_i := |A_i|$ and identify each $\alpha_i \in \Delta A_i$ with the tuple

$$(\alpha_i(a_1^i), \ldots, \alpha_i(a_{m_i}^i)) \in [0,1]^{m_i} \subseteq \mathbb{R}^{m_i}$$

where $A_i = \{a_1^i, \ldots, a_{m_i}^i\}$ $(1, 0, \ldots, 0)$

Thus, ΔA_i becomes a non-empty subset of $\mathbb{R}^{m_i}$, which is convex and compact.

Then show that the expected utility functions U_i are quasi-concave ~~and~~ on A_i and continuous. Thus all conditions of the theorem in 2.7. are satisfied.

$\boxed{q.e.d.}$

References

Holler, Manfred J. & Gerhard Illing,
Einführung in die Spieltheorie,
Springer-Verlag, Berlin, 5. Aufl., 2002.

Nisan, Noam, Tim Roughgarden, Éva Tardos, & Vijay V. Vazirani (eds.),
Algorithmic Game Theory,
Cambridge University Press, 2007.

Osborne, Martin J. & Ariel Rubinstein,
A Course in Game Theory,
MIT Press, Cambridge, MA, 1994.