Introduction to Game Theory

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2.6. Zerosum Games and Nash Equilibria

Definition: A <u>zerosum game</u> is a finite 2 player strategic game

$$G = \langle \{1,2\}, (A_i)_{i=1,2}, (u_i)_{i=1,2} \rangle$$

such that for all profiles $a, b \in A$

$$u_1(a) + u_2(a) = 0.$$

Constant sum games: $\exists c \in \mathbb{R}$ $u_1(a) + u_2(a) = c$ for all $a \in A$.

Example:

	L	M	R	
T	8, -8	3, -3	-6, 6	← -6
M	2, -2	-1, 1	3, -3	← -1
B	-6, 6	4, -4	8, -8	← -6
	↓ -8	↓ -4	↓ -8	

Idea: Choose action that is best for "you" under the assumption that the other player wants to minimize your utility.

Definition.

Let G be ZSG. $x^* \in A_1$ is called a maximinimizer for player 1, if

$$\min_{y\in A_2} u_1(x^*,y) \ge \min_{y\in A_2} u_1(x,y) \quad \text{for each } x\in A_1$$

$y^* \in A_2$ is maximinimizer of player 2 if

$$\min_{x\in A_1} u_2(x,y^*) \ge \min_{x\in A_1} u_2(x,y) \quad \text{for each } y\in A_2.$$

Consider as an example:

	L	R	
T	1, -1	2, -2	← 1
D	-2, 2	-4, 4	← -4
	↓ -1	↓ -2	

Profile (T, L) is a NE and a pair of maximinimizers.

We show: If G has a NE then any profile a that is NE is a pair of maximinimizers.

Lemma: Let G be a ZSG. Then

$$\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$$

Proof: Obviously, for each real-valued function f, it holds

$$(1) \quad \min_z\left(-f(z)\right) = -\max_z\left(f(z)\right)$$

Thus, we obtain

$$(2) \quad -\min_{x\in A_1} u_2(x,y) \overset{(1)}{=} \max_{x\in A_1} -u_2(x,y) \overset{-u_2=u_1}{=} \max_{x\in A_1} u_1(x,y)$$

Thus,

$$\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) \overset{(1)}{=} -\min_{y\in A_2} -\left(\min_{x\in A_1} u_2(x,y)\right)$$

$$= -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y) \quad \blacksquare$$

Proposition: Let G be a ZSG.

(a) Whenever (x^*, y^*) is a NE of G, then x^* and y^* are maximinimizers of player 1 and 2, respectively.

(b) If (x^*, y^*) is a NE of G, then

$$\max_{x \in \Delta_1} \min_{y \in \Delta_2} u_1(x,y)$$

$$= \min_{y \in \Delta_2} \max_{x \in \Delta_1} u_1(x,y)$$

$$= u_1(x^*, y^*) \leftarrow -u_2(x^*, y^*)$$

Thus all NE of G have the same payoff.

(c) If $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ and x^* and y^* are maximinimizers of player 1 and 2, respectively, then (x^*, y^*) is a NE. In particular, if (x_1^*, y_1^*) and (x_2^*, y_2^*) are NE of G, then so are (x_1^*, y_2^*) and (x_2^*, y_1^*).

Proof:

(a) + (b). Let (x^*, y^*) be a NE of G.

By defn. of NE,

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for each } y \in A_2$$

From $u_2 = -u_1$ it follows

$$u_1(x^*, y^*) \leq u_1(x^*, y), \quad \text{for each } y \in A_2.$$

and hence

$$u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y)$$

$$(1) \qquad \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y).$$

Moreover by def of NE,

$$u_1(x^*, y^*) \geq u_1(x, y^*), \text{ for each } x \in A_1.$$

and hence,

$$u_1(x^*, y^*) \geq \min_{y \in A_2} u_1(x, y) \qquad \text{for each } x \in A_1$$

and thus,

$$(2) \qquad u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y).$$

From (1) and (2) we obtain:

$$(3) \quad u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x,y)$$

Thus by (1),

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_y u_1(x,y) \quad \text{for each } x \in A_1.$$

i.e. x^* is a maximinimizer of player 1.

In a similar way, prove that y^* is a maximinimizer of player 2 and that

$$(4) \quad u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$$

This shows (a).

Moreover, we have:

$$u_1(x^*, y^*) \overset{(3)}{=} \max_{x \in A_1} \min_{y \in A_2} u_1(x,y)$$

$$\overset{-u_1 = u_2 + (4)}{=} - \max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$$

$$\overset{\text{Lemma}}{=} \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$$

This shows (b).

References

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