Constraint Satisfaction Problems

Qualitative Representation and Reasoning

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1 Motivation

- Qualitative Constraint Satisfaction Problems

Quantitative vs. qualitative representations

Spatio-temporal configurations can be described quantitatively by specifying the coordinates of the relevant objects:

Example: At time point 10.0 object A is at position $(11.0, 1.0, 23.7)$, at time point 11.0 at position $(15.2, 3.5, 23.7)$. From time point 0.0 to 11.0, object B is at position $(15.2, 3.5, 23.7)$. Object C is at time point 11.0 at position $(300.9, 25.6, 200.0)$ and at time point 35.0 at $(11.0, 1.0, 23.7)$.

Often, however, a qualitative description (using a finite vocabulary) is more adequate:

Example: Object A hit object B. Afterwards, object C arrived.

Sometimes we want to reason with such descriptions.

Example: Object C was not close to object A, when it hit object B.

Representation of qualitative knowledge

Intention: describe configurations in an infinite (continuous) domain using a finite vocabulary and reason about these descriptions

- Specification of a vocabulary: usually a finite set of relations (often binary) that are pairwise disjoint and jointly exhaustive
- Specification of a language: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal semantics
- Analysis of computational properties and design of reasoning methods (often constraint propagation)
- Perhaps, specification of operational semantics for verifying whether a relation holds in a given quantitative configuration

Applications in ...

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- ...

Example: Qualitative temporal relations

Suppose, we want to talk about time instants (points) and binary relations over them.

- Vocabulary: $X = Y$ (X equals Y), $X < Y$ (X before Y), and $X > Y$ (X after Y).
- Language:
  - Allow for disjunctions of basic relations to express indefinite information. Use unions of relations to express that. For instance, $< \cup =$ expresses $\leq$.
  - 2^3 different relations (including the impossible and the universal relation)
  - Use sets of atomic formulae with these relations to describe configurations. For example:

$$\{x = y, y\,(< \cup >)\,z\}$$

- Semantics: Interpret the time point symbols and relation symbols over the real (or rational) numbers.

Some reasoning problems

$$\left\{x\,(< \cup =)\,y, y\,(< \cup =)\,z, v\,(< \cup =)\,y, w > y, z\,(< \cup =)\,x\right\}$$

- Satisfiability: Are there values for all time points such that all formulae are satisfied?
- Satisfiability with $v = w$?
- Finding a satisfying instantiation of all time points
- Deduction: Does $x\{=\}y$ follow logically?
  Does $v \leq w$ follow?
- Finding a minimal description: What are the most constrained relations that describe the same set of instantiations?

From a logical point of view . . .

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in satisfiability, model finding, and deduction

Let $\mathcal{B}$ be a finite set of (binary) relations on some (infinite) domain D (elements of $\mathcal{B}$ are called base relations).
We require:

- The relations in $\mathcal{B}$ are JEPD, i.e., jointly exhaustive and pairwise disjoint.
- $\mathcal{B}$ is closed under converses.

Then:

- Let $\mathcal{A}$ be the set of relations that can be built by taking the unions of relations from $\mathcal{B}$ ($\rightsquigarrow 2^{|\mathcal{B}|}$ different relations).
- $\mathcal{A}$ is closed under converse, complement, intersection and union.
- Often, $\mathcal{A}$ is closed under composition of base relations, i.e., for all $B, B' \in \mathcal{B}$,
  $$B \circ B' \in \mathcal{A}.$$
  Then, $\mathcal{A}$ is closed under composition of arbitrary relations.
  But often this condition is not satisfied.

Computing operations on relations

Let $\mathcal{A}$ be the system of relations over a set of base relations $\mathcal{B}$ that satisfies all the conditions above.
We may write relations as *sets* of base relations:

$$B_1 \cup \cdots \cup B_n \cong \{B_1, \ldots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \ldots B_n\} \circ \{B'_1, \ldots, B'_m\} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_i \circ B'_j$$

Converse:

$$\{B_1, \ldots, B_n\}^{-1} = \{B_1^{-1}, \ldots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \ldots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and union are defined in the usual set-theoretical way.

Reasoning problems

Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- Given in addition xRy: Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair x, y of variables the strongest constrained (minimal) relation entailed by the CSP.

Reductions between CSP problems

Theorem
CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.
CSAT $\leq_T$ CENT and CENT $\leq_T$ CMIN are obvious.

CENT $\leq_T$ CSAT: We solve CENT ($CSP \models xRy$?) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations. Let $B_1, \ldots, B_k$ be the relations for which we get a positive answer. Then $x\{B_1, \ldots, B_k\}y$ is entailed by the CSP.

CMIN $\leq_T$ CENT: We use entailment for computing the minimal constraint for each pair of variables. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. □

The Path Consistency Method

Given a qualitative CSP with $R_{v_1,v_2} = R^{-1}_{v_2,v_1}$. Then the path consistency method is to apply the operation

$$R_{v_1,v_2} \leftarrow R_{v_1,v_2} \cap (R_{v_1,v_3} \circ R_{v_3,v_2}).$$

on all the constraints of the network until a fixpoint is reached.

The path consistency method guarantees ...

- sometimes minimality
- sometimes satisfiability
- however sometimes the CSP is not satisfiable, even if the CSP contains only base relations

Example: Point relations

Composition table:

	$<$	$=$	$>$
$<$	$<$	$<$	$<,=,>$
$=$	$<$	$=$	$>$
$>$	$<,=,>$	$>$	$>$

Figure: Composition table for the point algebra. For example: $\{<\} \circ \{=\} = \{<\}$

- $\{<,=\} \circ \{<\} = \{<\}$
- $\{<,>\} \circ \{<\} = \{<,=,>\}$
- $\{<,=\}^{-1} = \{>,=\}$
- $\{<,=\} \cap \{>,=\} = \{=\}$

Some properties of the point relations

Theorem
A path consistent CSP over the point relations is satisfiable.

In particular, the path consistency method decides satisfiability.

Theorem
A path consistent CSP over all point relations without $\{<,>\}$ is minimal.

Proofs later ...

2 Qualitative Constraint Languages

- Constraint Propagation

Qualitative constraint languages

From now on, let D be a finite or infinite domain.

Definition
A partition scheme on D is any non-empty, finite set Δ of binary relations on D such that:

- Δ defines a partition of $D \times D$.
- Δ contains the binary identity relation id_D.
- Δ is closed under converses.

Definition
A constraint language of binary relations on D, Γ, is said to be generated from a partition scheme Δ, if Γ consists of all finite unions of relations in Δ.

Constraint languages in this sense will be referred to as qualitative constraint languages.

Qualitative constraint network

Let Γ be a subset of a qualitative constraint language with partition scheme Δ.

Definition
A qualitative constraint network over Γ is a triple

$$P = \langle V, D, C \rangle ,$$

where:

- V is a non-empty and finite set of variables,
- D is an arbitrary non-empty set (domain),
- C is a finite set of constraints $C_1, \ldots, C_q$, i.e., each constraint C_i is a pair (s_i, R_i), where s_i is a pair of variables and R_i is a binary relation contained in Γ.

Weak composition

Let Γ be a qualitative constraint language with partition scheme Δ. For $R, S \in \Gamma$, define:

$$R \circ_w S := \bigcup \{T \in \Delta : T \cap (R \circ S) \neq \emptyset\}$$

— $\circ_w$ is called weak composition of R and S.

Lemma
For all relations $R, S, T \in \Gamma$,

- $R \circ S \subseteq R \circ_w S$;
- $T \cap (R \circ S) = \emptyset$ *if and only if* $T \cap (R \circ_w S) = \emptyset$;
- $(R \circ_w S)^{-1} = S^{-1} \circ_w R^{-1}$;
- $R \circ_w (S \cup T) = (R \circ_w S) \cup (R \circ_w T)$.

Weak composition: Examples

Example:
Consider a linear order on a domain with 2 elements $a < b$. The relations $R_<, R_=, R_>$ define a partition schema on D. It holds:

$$R_< \circ R_< = R_> \circ R_> = \emptyset, \;\; R_< \circ R_> = \{(a,a)\}, \;\; R_> \circ R_< = \{(b,b)\}$$

but

$$R_< \circ_w R_< = R_> \circ_w R_> = \emptyset, \;\; R_< \circ_w R_> = R_=, \;\; R_> \circ_w R_< = R_=$$

Moreover,

$$(R_< \circ_w R_>) \circ_w R_> = R_= \circ_w R_> = R_> \neq \emptyset = R_< \circ_w \emptyset = R_< \circ_w (R_> \circ_w R_>).$$

Example:
Consider a linear order on a domain with 3 elements $a < b < c$. Then

$$R_< \circ R_< = \{(a,c)\} \quad \text{but} \quad R_< \circ_w R_< = R_<.$$

Qualitative languages and algebras

Let Γ be a qualitative constraint language with partition scheme Δ. As spelled out before, each relation R in Γ can be represented by a finite disjunction of "base relations" $B_1, \ldots, B_k \in \Delta$. In what follows we identify R with the set of its base relations

$$\{B_1, \ldots, B_k\}.$$

Lemma
For each partition scheme Δ, the tuple

$$\left\langle 2^\Delta, \cap, \cup, \circ_w, \mathbf{C}_\Delta, {}^{-1}, \emptyset, \Delta, \mathrm{id}_\Delta \right\rangle$$

defines a non-associative relation algebra.

Algebraically closed networks

A qualitative network $P = \langle V, D, C\rangle$ is normalized, if

- for each pair of variables x, y, C contains at least one constraint $((x,y),R)$;
- for each constraint $((x,x),R)$ in C, $R = \mathrm{id}_D$;
- for constraints $((x,y),R)$ and $((y,x),S)$ in C, $R = S^{-1}$.

In what follows we will always assume that constraint networks are normalized.

Definition
A qualitative constraint network P is algebraically closed (or: a-closed), if for all constraints $((x,y),R)$, $((x,z),S)$, and $((z,y),T)$ of P, it holds:

$$R \subseteq S \circ_w T.$$

Note: If P is algebraically closed, then $R = R \cap (S \circ_w T)$.

Constraint propagation

The path consistency algorithm can only be used if the underlying partition scheme is closed under composition, i.e., if for each pair of relations $R, S \in \Delta$, $R \circ S$ is a (finite) union of a subset of Δ.
The algebraic closure algorithm is a variant of the path consistency algorithm. Instead of ordinary composition of relations, we use weak composition.
Since weak composition is an upper approximation of composition only, the algebraic closure algorithm may not result in a path-consistent network.

Let $P = \langle V, D, C\rangle$ be a (normalized) qualitative constraint network.
Let $Table[i,j]$ be a $n \times n$-matrix (n: number of variables), in which we record the constraints between the variables.

Algebraic closure algorithm

EnforceAlgClosure (P):
Input: a qualitative network $P = \langle V, D, C\rangle$
Output: "inconsistent", or an equivalent algebraically closed network P'

$$Paths(i,j) = \{(i,j,k) : 1 \leq k \leq n, k \neq i,j\} \cup \{(k,i,j) : 1 \leq k \leq n, k \neq i,j\}$$

$Queue := \bigcup_{i,j} Paths(i,j)$
while $Q \neq \emptyset$
  select and delete (i,k,j) from Q
  $T := Table[i,j] \cap (Table[i,k] \circ_w Table[k,j])$
  if $T = \emptyset$
    return "inconsistent"
  elseif $T \neq Table[i,j]$
    $Table[i,j] := T$
    $Table[j,i] := T^{-1}$
    $Queue := Queue \cup Paths(i,j)$
return P' with the refined constraints as recorded in *Table*

Computing on the symbolic level

Let Γ be a qualitative constraint language with partition scheme Δ. We suppose that we have determined (by some formal proof or some computation) the (weak) composition table for Δ, i.e.,

$$\circ_{(w)} : \Delta \times \Delta \to 2^{\Delta}.$$

Let now B be a finite set of symbols (bijective with Δ). Then 2^B is a Boolean algebra, from which we obtain a (non-associative) relation algebra, if we extend $\circ_{(w)}$ to a function

$$\circ_{(w)} : 2^B \times 2^B \to 2^B.$$

Now we can perform all the operations needed in the path consistency/a-closure algorithm on the symbolic level.

3 Allen's Interval Algebra

Allen's Interval Calculus

- Allen's interval calculus (IA): time intervals and binary relations over them
- Let $\langle \mathbb{R}, < \rangle$ be the linear order on the real numbers (conceived of as the flow of time).
  Then, the domain D of Allen's calculus is the set of all *intervals*

  $$X = (X^-, X^+) \in \mathbb{R}^2, \text{ where } X^- < X^+$$

  (naïve approach)
- Relations between concrete intervals, e. g.:
  (1.0,2.0) strictly before (3.0,5.5)
  (1.0,3.0) meets (3.0,5.5)
  (1.0,4.0) overlaps (3.0,5.5)
  ...

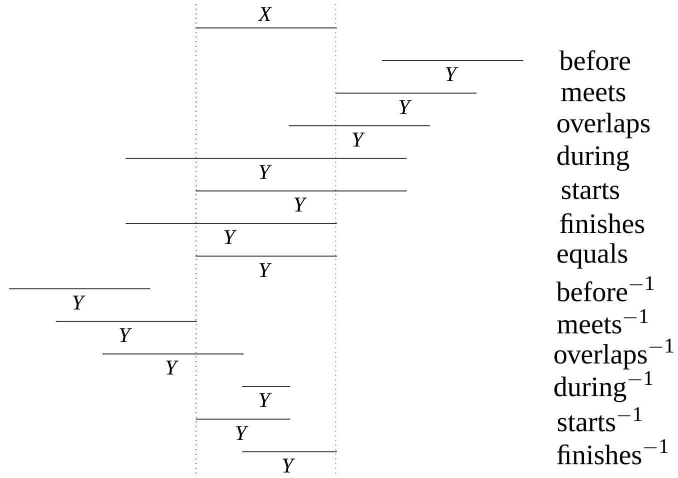
IA: The base relations

To determine all possible relation between Allen intervals, we determine how one can order the four points of two intervals:

Relation	Symbol	Name
$\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$	$\prec$	before
$\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$	m	meets
$\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$	o	overlaps
$\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$	s	starts
$\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$	f	finishes
$\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$	d	during
$\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the *converse* relations (obtained by exchanging X and Y)

IA: The 13 base relations graphically



IA: Partition scheme and composition

Lemma

The 13 base relations of Allen's interval calculus define a partition scheme on the set of all Allen intervals.

In what follows:

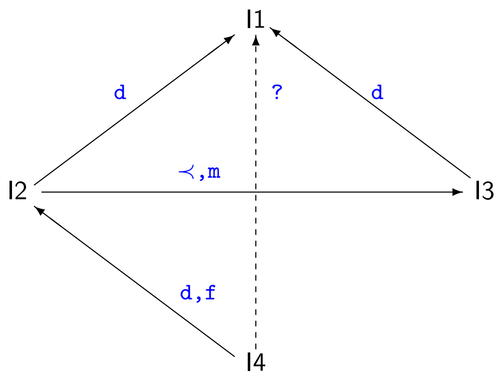
- IA: the qualitative constraint language generated from all base relations of Allen's interval calculus (contains $2^{13} = 8192$ relations)
- IA-$\mathcal{B}$: the subclass of IA containing base relations only

Lemma

The set of base relations of Allen's interval calculus is closed under composition.

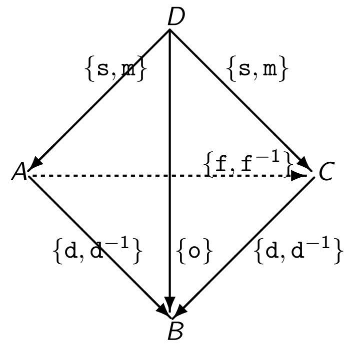
	$\prec$	$\succ$	d	d^{-1}	o	o^{-1}	m	m^{-1}	s	s^{-1}	f	f^{-1}
$\prec$	$\prec$	$\mathbf{B}$	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$
$\succ$	$\mathbf{B}$	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$	$\succ$
d	$\prec$	$\succ$	d	$\mathbf{B}$	$\prec$ o m d s	$\succ$ o^{-1} m^{-1} d f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d f	d	$\prec$ o m d s
d^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	$\mathbf{B}-$ $\prec\succ$ m m^{-1}	d^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	d^{-1}	o^{-1} d^{-1} s^{-1}	d^{-1}
o	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	$\mathbf{B}-$ $\prec\succ$ m m^{-1}	$\prec$	o^{-1} d^{-1} s^{-1}	o	d^{-1} f^{-1} o	d s o	$\prec$ o m
o^{-1}	$\prec$ o m, d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$, o^{-1} $m^{-1}$$d^{-1}$ s^{-1}	$\mathbf{B}-$ $\prec\succ$ m m^{-1}	$\succ$ o^{-1} m^{-1}	o d^{-1} f^{-1}	$\succ$	o^{-1} d f	o^{-1} $\succ$ m^{-1}	o^{-1}	o^{-1} d^{-1} s^{-1}
m	$\prec$	$\succ$ o^{-1} $m^{-1}$$d^{-1}$ s^{-1}	o d s	$\prec$	$\prec$	o d s	$\prec$	f f^{-1} $\equiv$	m	m	d s o	$\prec$
m^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$	o^{-1} d f	$\succ$	s s^{-1} $\equiv$	$\succ$	d f o^{-1}	$\succ$	m^{-1}	m^{-1}
s	$\prec$	$\succ$	d	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	o^{-1} d f	$\prec$	m^{-1}	s	s s^{-1} $\equiv$	d	$\prec$ m o
s^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} .d f	d^{-1}	o d^{-1} f^{-1}	o^{-1}	o d^{-1} f^{-1}	m^{-1}	s s^{-1} $\equiv$	s^{-1}	o^{-1}	d^{-1}
f	$\prec$	$\succ$	d	$\succ$ o^{-1} $m^{-1}$$d^{-1}$ s^{-1}	o d s	$\succ$ o^{-1} m^{-1}	m	$\succ$	d	$\succ$ o^{-1} m^{-1}	f	f f^{-1} $\equiv$
f^{-1}	$\prec$	$\succ$ o^{-1} $m^{-1}$$d^{-1}$ s^{-1}	o d s	d^{-1}	o	o^{-1} d^{-1} s^{-1}	m	s^{-1} o^{-1} d^{-1}	o	d^{-1}	f f^{-1} $\equiv$	f^{-1}

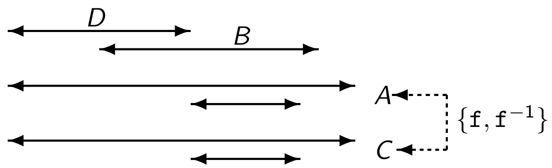
IA: An example



Compose the constraints: I4 $\{\mathtt{d},\mathtt{f}\}$ I2 and I2 $\{\mathtt{d}\}$ I1: I4 $\{\mathtt{d}\}$ I1.

IA: Example for incompleteness





IA: NP-hardness

Theorem (Kautz & Vilain)

Deciding satisfiability over IA is NP-hard.

Proof.

Reduction from 3-colorability (the original proof uses 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability.
Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

$$\begin{array}{cccl} 1 & \{\mathtt{m}\} & 2 & \\ 2 & \{\mathtt{m}\} & 3 & \\ v_i & \{\mathtt{m}, \equiv, \mathtt{m}^{-1}\} & 2 & \forall v_i \in V \\ v_i & \{\mathtt{m}, \mathtt{m}^{-1}, \prec, \succ\} & v_j & \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable *iff* G can be colored with 3 colors. □

IA: Clause representation

Following, we will look at polynomial special cases, i.e., subclasses of the qualitative constraint language IA.

For this we start from a natural translation of interval relations/constraints (of the form $X\ R\ Y$) into clause formulas over atoms of the form $a\ op\ b$, where:

- $a, b \in \{X^-, X^+, Y^-, Y^+\}$; and
- $op \in \{<, >, =, \leq, \geq\}$.

Example: All base relations can be expressed as unit clauses.

Lemma

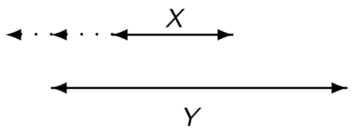
Let P be a constraint network over IA, and let $\pi(P)$ be the translation of P into clause form.
P is satisfiable iff $\pi(P)$ is satisfiable over the real numbers.

IA: The Continuous Endpoint Class

Continuous Endpoint Class IA-$\mathcal{C}$: the subset of IA consisting of those relations with a clause form containing only unit clauses, where $\neg(a = b)$ is forbidden.

Example: All basic relations and, e.g., $\{\mathtt{d}, \mathtt{o}, \mathtt{s}\}$, because

$$\begin{aligned}\pi(X\ \{\mathtt{d},\mathtt{o},\mathtt{s}\}\ Y) \quad = \quad & \{X^- < X^+, Y^- < Y^+,\\ & X^- < Y^+, X^+ > Y^-,\\ & X^+ < Y^+\}\end{aligned}$$



The set IA-$\mathcal{C}$ contains 83 relations. It is closed under intersection, composition, and converses (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

IA: Consistency for IA-$\mathcal{C}$

One can prove:

Lemma

Each 3-consistent interval CSP over IA-$\mathcal{C}$ is globally consistent.

From this we can conclude:

Theorem (van Beek)

Applied to networks over IA-$\mathcal{C}$, enforcing path consistency decides satisfiability and solves the minimal label problem.

Corollary

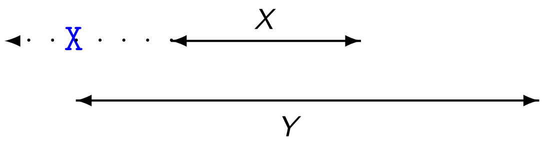
A path-consistent interval constraint network containing base relations only is satisfiable.

IA: The Endpoint Subclass

Endpoint Subclass: IA-$\mathcal{P}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is now allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\begin{aligned}\pi(X\ \{\mathtt{d},\mathtt{o}\}\ Y) \quad = \quad & \{X^- < X^+, Y^- < Y^+,\\ & X^- < Y^+, X^+ > Y^-, X^- \neq Y^-,\\ & X^+ < Y^+\}\end{aligned}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

The path consistency method decides satisfiability over IA-$\mathcal{P}$.

IA: The ORD-Horn Subclass

ORD-Horn Subclass: IA-$\mathcal{H}$ is the subclass of IA that permits a clause form containing only Horn clauses, where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in$ IA-$\mathcal{P}$ and $\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}$:

$$\pi(X\{\mathtt{o},\mathtt{s},\mathtt{f}^{-1}\}Y) = \begin{Bmatrix} X^- \leq X^+, X^- \neq X^+,\\ Y^- \leq Y^+, Y^- \neq Y^+,\\ X^- \leq Y^-,\\ X^- \leq Y^+, X^- \neq Y^+,\\ Y^- \leq X^+, X^+ \neq Y^-,\\ X^+ \leq Y^+,\\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{Bmatrix}.$$

Partial orders: The *ORD* Theory

Let *ORD* be the following theory:

$$\begin{array}{llcll} \forall x,y,z: & x \leq y \wedge y \leq z & \rightarrow & x \leq z & \textit{(transitivity)} \\ \forall x: & x \leq x & & & \textit{(reflexivity)} \\ \forall x,y: & x \leq y \wedge y \leq x & \rightarrow & x = y & \textit{(anti-symmetry)} \\ \forall x,y: & x = y & \rightarrow & x \leq y & \textit{(weakening of =)} \\ \forall x,y: & x = y & \rightarrow & y \leq x & \textit{(weakening of =).} \end{array}$$

- *ORD* describes partially ordered sets, $\leq$ being the ordering relation.
- *ORD* is a Horn theory
- What is missing wrt. *dense* and *linear* orders?

Satisfiability over partial orders

Lemma
Let Θ be a CSP over IA-$\mathcal{H}$. Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.
$\Rightarrow$: Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.
$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

Complexity of CSAT(IA-$\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Lemma
$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Theorem
CSAT(IA-$\mathcal{H}$) can be decided in polynomial time.

Proof.
CSAT(IA-$\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. □

IA-$\mathcal{C}$ $\subset$ IA-$\mathcal{P}$ $\subset$ IA-$\mathcal{H}$ with $|\text{IA-}\mathcal{C}| = 83$, $|\text{IA-}\mathcal{P}| = 188$, $|\text{IA-}\mathcal{H}| = 868$

Path consistency and the OH-class

Lemma
Let Θ be a path-consistent set over IA-$\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \quad \textit{iff} \quad \Theta \textit{ is satisfiable}$$

Proof idea: One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem
Enforcing path consistency decides CSAT(IA-$\mathcal{H}$).

- $\rightsquigarrow$ Maximality of IA-$\mathcal{H}$?
- $\rightsquigarrow$ Do we have to check all 8192 − 868 extensions?

IA: The ORD-Horn subclass: Maximality

A computer-aided case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras containing all base relations that strictly contain IA-$\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{\mathtt{d}, \mathtt{d}^{-1}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}\} \in \mathcal{X}_1$$
$$N_2 = \{\mathtt{d}^{-1}, \mathtt{o}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}^{-1}\} \in \mathcal{X}_2$$

The clause forms of these relations contain "proper" disjunctions!

Theorem

The satisfiability problem over IA-$\mathcal{H} \cup \{N_i\}$ is NP-complete.

Lemma

IA-$\mathcal{H}$ is the only maximal tractable subclass that contains all base relations of IA.

IA: Solving general Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Method works on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

$\rightsquigarrow$ Which tractable fragment should one use?

IA: Branching factors

- If the labels are split into base relations, then on average a label is split into

  6.5 relations

- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into

  2.955 relations

- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into

  2.533 relations

$\rightsquigarrow$ A difference of 0.422 which becomes significant, when applied to extremely hard instances

Summary

- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path consistency method.
- Can be used in practice for backtracking algorithms.

Outlook

- Qualitative representation and reasoning usually starts with a finite vocabulary (a finite set of relations).
- Qualitative descriptions are usually simply logical theories consisting of sets of atomic formulae (and some background theory).
- Reasoning problems are (as usual) satisfiability, model finding, and deduction.
- Can be addressed with CSP methods (but note: infinite domains).
- Path consistency is the basic reasoning step . . . sometimes this is enough.
- Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.

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