Constraint Satisfaction Problems

Constraint Networks

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May 7, 2012

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1 Constraint Networks

- Constraint networks
- Solution
- Normalized Constraint Networks
- Deduction
- Minimal Network

Constraint networks

Definition

A constraint network is a triple

$$N = \langle V, \mathrm{dom}, C \rangle$$

where:

- V is a non-empty and finite set of variables;
- dom is a function that assigns to each variable $v \in V$ a non-empty set $\mathrm{dom}(v)$ ($\mathrm{dom}(v)$ is called the domain of v, elements of $\mathrm{dom}(v)$ are called values);
- C is a set of relations over variables of V (called constraints), i.e., each constraint is a relation $R_{x_1,\ldots,x_m}$ over some scheme $S = (x_1, \ldots, x_m)$ of variables in V.

The set of constraint schemes $\{S_1, \ldots S_t\}$ is called network scheme.

Constraint networks

If we assume an ordering of the variables in V, we can write networks more compactly:

Definition

A constraint network is a triple

$$N = \langle V, D, C \rangle$$

where:

- $V = (v_1, \ldots, v_n)$ is a non-empty and finite sequence of variables;
- $D = (D_1, \ldots, D_n)$ is a sequence of domains for V (D_i is the domain of variable v_i);
- C is a set of constraints $R_{\overline{x}}$ where $\overline{x} = (v_{i_1}, \ldots, v_{i_m})$ is a scheme of variables in V and $R \subseteq D_{i_1} \times \cdots \times D_{i_m}$.

Example: 4-queens problem

The 4-queens problem can be represented as single constraint network.
For example, consider variables $v_1, \dots, v_4$ (each associated to a column of the 4×4-chess board).
Each variable v_i has as its domain $D_i = \{1, \dots, 4\}$ (conceived of as the row positions of a queen in column i).

	v_1	v_2	v_3	v_4
1				
2				
3				
4				

Define then binary constraints (thus encoding "non-attacking queen positions"):

$$R_{v_1,v_2} := \{(1,3),(1,4),(2,4),(3,1),(4,1),(4,2)\}$$

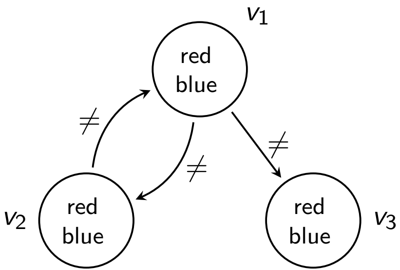
$$R_{v_1,v_3} := \{(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)\}$$

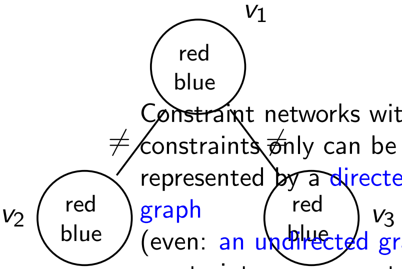
...

Example: Graph colorability

k-Colorability of a graph G can be represented as a constraint network of the following form:

$$V = \{v_i : v_i \text{ is a vertex in } G\}$$
$$D_i = \{1, \ldots, k\} \ (v_i \in V)$$
$$C = \{((v_i, v_j), \neq) : \{v_i, v_j\} \text{ is an edge of } G\}$$





Constraint networks with binary constraints only can be represented by a directed labeled graph (even: an undirected graph if all constraints are symmetric).

Solution of a constraint network

Definition

A solution of a constraint network $N = \langle V, D, C \rangle$ is a (variable) assignment

$$a\colon V \to \bigcup_{i\colon v_i \in V} D_i$$

such that

(a) $a(v_i) \in D_i$, for each $v_i \in V$,

(b) $(a(x_1), \ldots, a(x_m)) \in R$ for each constraints $R_{x_1,\ldots,x_m}$ in C.

N is called solvable (or: satisfiable) if N has a solution.

$\mathrm{Sol}(N)$ denotes the set of all solutions of N.

Instantiation, partial solution

Let $N = \langle V, D, C \rangle$ be a constraint network.

Definition

(a) An instantiation of a subset V' of V is an assignment $a : V' \to \bigcup_{i:\, v_i \in V'} D_i$ with $a(v_i) \in D_i$.

(b) An instantiation a of V' is called partial solution if a satisfies each constraint R_S in C with $S \subseteq V'$.
We also say: a is consistent relative to N.

(c) For an instantiation a of a subset $V' = \{x_1, \dots, x_m\}$ and a constraint R_S with scope $S \subseteq V'$, let

$$\overline{a}[S] := (a(x_1), \dots, a(x_m)).$$

Hence a solution is an instantiation of all variables in V that is consistent relative to N.

Instantiation, solution

Note:

(a) An instantiation of variables in $V' \subseteq V$, a, is a partial solution (consistent relative to N) iff

$$\overline{a}[S] \in R, \quad \text{for each constraint } R \text{ with scope } S \subseteq V'.$$

(b) Not every partial solution is part of a (full) solution, i.e., there may be partial solutions of a constraint network that cannot be extended to a solution. For the 4-queens problem, for example,

	v_1	v_2	v_3	v_4
1	q			
2			q	
3				
4		q		

Normalized constraint network

Let $N = \langle V, D, C\rangle$ be a constraint network.
Due to our definition it is possible that C contains constraints

$$R_{v_{i_1},\dots,v_{i_k}} \quad \text{and} \quad S_{v_{j_1},\dots,v_{j_k}}$$

where $(j_1, \dots, j_k)$ is just a permutation of $(i_1, \dots, i_k)$.

Without changing the set of solutions, we can simplify the network by deleting $S_{v_{j_1},\dots,v_{j_k}}$ from C and rewriting $R_{v_{i_1},\dots,v_{i_k}}$ as follows:

$$R_{v_{i_1},\dots,v_{i_k}} \leftarrow R_{v_{i_1},\dots,v_{i_k}} \cap \pi_{v_{i_1},\dots,v_{i_k}}(S_{v_{j_1},\dots,v_{j_k}}).$$

Given a fixed order on the set of variables V, we can systematically delete-and-refine constraints. This results in a constraint network that contains at most one constraint for each subset of variables. Such a network is called a normalized constraint network.

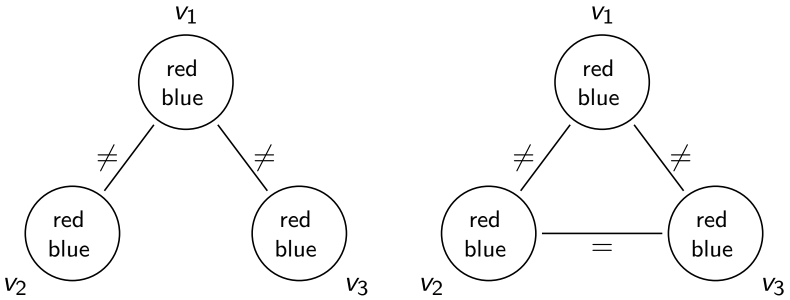
Equivalence

Let N and N' be constraint networks on the same set of variables and on the same domains for each variable.

Definition

N and N' are called equivalent if they have the same set of solutions.

Example:



Tightness

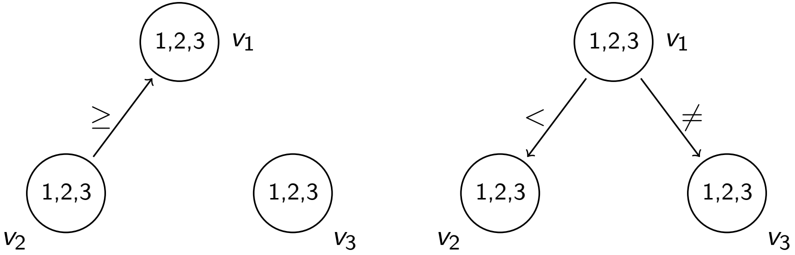
Let N and N' be (normalized) constraint networks on the same set of variables and on the same domains for each variable.

Definition

N is as tight as N' if for each constraint R_S of N,

(a) N' has no constraint with the same scope as R_S, or

(b) $R \subseteq \pi_S(R'_{S'})$, where $R'_{S'}$ is the constraint of N' with the same scope as R_S.



Intersection of networks

Definition

The intersection of N and N', $N \cap N'$, is the network defined by intersecting for each scope the constraints $R_S \in C$ and $R'_{S'} \in C'$ with the same scope, i.e., modulo a suitable permutation of the constraint schemes,

$$R''_S := R_S \cap R'_S.$$

If for a scope S only one of the networks contains a constraint, then we set:

$$R''_S := R_S \quad (\text{or } := R'_S, \text{ resp.})$$

Lemma

If N and N' are equivalent networks, then $N \cap N'$ is equivalent to both networks and as tight as both networks.

Minimal network

Definition

Let N_0 be a constraint network and let $N_1, \ldots, N_k$ be the set of all constraint networks (defined on the same set of variables and the same domains) that are equivalent to N_0.

$$\bigcap_{1 \leq i \leq k} N_i$$

is called the minimal network of N_0.

Lemma

The minimal network is equivalent to and as tight as all the constraint networks N_i. There is no network equivalent to N_0 that is tighter than the minimal network.

2 Projection Networks

Projecting constraints

Let R_S be a constraint with scheme $S = (x_1, \dots, x_m)$ (we can think of R_S as a constraint network ...).

Definition

The projection network of R_S, $\mathrm{Proj}(R_S)$, is the constraint network defined by:

$$V := S, \;\; D_i := \pi_{x_i}(R_S), \;\; R'_{x_i,x_j} := \pi_{x_i,x_j}(R_S)$$

for all variables x_i and variable pairs x_i, x_j.

Consider $R_{x,y,z}$ with $R = \{(a,a,b),(a,b,b),(a,b,a)\}$.
Then $\mathrm{Proj}(R_{x,y,z})$ consists of the following constraints: $R'_{x,y} = \{(a,a),(a,b)\}$,
$R'_{x,z} = \{(a,b),(a,a)\}$, and
$R'_{y,z} = \{(a,b),(b,b),(b,a)\}$.

In this case: $\mathrm{Sol}(\mathrm{Proj}(R_{x,y,z})) = R_{x,y,z}$.

Projecting constraints

The projection network is an upper approximation by binary networks in the following sense:

Lemma

Any solution of R_S (as a network) defines a solution of $\mathrm{Proj}(R_S)$, i.e.,

$$R_S \subseteq \mathrm{Sol}(\mathrm{Proj}(R_S)).$$

□

Lemma

$\mathrm{Proj}(R_S)$ is the "tightest" upper approximation of R_S by binary constraint networks, i.e., there is no binary constraint network N' defined on the variables of R_S such that:

$$R \subseteq \mathrm{Sol}(N') \subsetneq \mathrm{Sol}(\mathrm{Proj}(R_S)).$$

Binary representation

Definition

A relation R_S with scope S has a binary representation if the relation (conceived of as a network) is equivalent to $\mathrm{Proj}(R_S)$.

From the fact that a relation has a binary representation, it does not follow that all its projections have binary representations as well (Exercise!).

Definition

A relation R_S with scope S is binary decomposable if the relation itself and all its projections to subsets of S (with at least 3 elements) have a binary representation.

3 Constraint Networks and Graphs

- Primal Constraint Graphs
- Dual Constraint Graph
- Constraint Hypergraph

Primal constraint graphs

Let $N = \langle V, D, C \rangle$ be a (normalized) constraint network.

Definition

The primal constraint graph of a network $N = \langle V, D, C \rangle$ is the undirected graph

$$G_N := \langle V, E_N \rangle$$

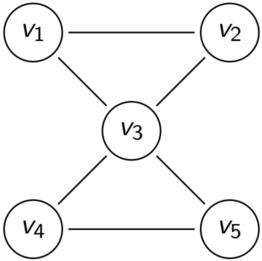
where

$$\{u, v\} \in E_N \iff \{u, v\} \text{ is a subset of the scope of some constraint in } N.$$

Primal constraint graph: Example

Consider a constraint network with variables $v_1, \ldots, v_5$ and two ternary constraints R_{v_1,v_2,v_3} and S_{v_3,v_4,v_5}.

Then the primal constraint graph of the network has the form:



Absence of an edge between two variables/nodes means that there is no explicit constraint in which both variables participate.

Dual constraint graphs

Definition

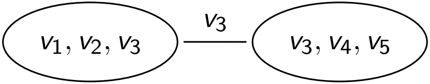
The dual constraint graph of a constraint network $N = \langle V, D, C \rangle$ is the labeled graph

$$D_N := \langle V', E_N, l \rangle$$

with

$$X \in V' \iff X \text{ is the scope of some constraint in } N$$
$$\{X, Y\} \in E_N \iff X \cap Y \neq \emptyset$$
$$l : E_N \to 2^V, \quad \{X, Y\} \mapsto X \cap Y$$

In the example above, the dual constraint graph is:



Constraint hypergraph

Definition

The constraint hypergraph of a constraint network $N = \langle V, D, C \rangle$ is the hypergraph

$$H_N := \langle V, E_N \rangle$$

with

$$X \in E_N \iff X \text{ is the scope of some constraint in } N.$$

In the example above (constraint network with variables $v_1, \ldots, v_5$ and two ternary constraints R_{v_1,v_2,v_3} and S_{v_3,v_4,v_5}) the hyperedges of the constraint hypergraph are:

$$E_N = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}\}.$$

4 Solving Constraint Networks

Simple solution strategy: Backtracking search

Backtracking: search systematically for consistent partial instantiations in a depth-first manner:

- forward phase: extend the current partial solution by assigning a consistent value to some new variable (if possible)
- backward phase: if no consistent instantiation for the current variable exists, we return to the previous variable.

Backtracking algorithm

Backtracking(N, a):

Input: a constraint network $N = \langle V, D, C \rangle$ and
a partial assignment a of N
(e.g., the empty instantiation $a = \{\,\}$)
Output: a solution of N or "inconsistent"

if a is not consistent with N:
    return "inconsistent"
if a is defined for all variables in V:
    return a
select some variable v_i for which a is not defined
for each value x from D_i:
    $a' := a \cup \{v_i \mapsto x\}$
    $a'' \leftarrow$ Backtracking(N, a')
    if a'' is not "inconsistent":
        return a''
return "inconsistent"

Literature

Rina Dechter.
Constraint Processing,
Chapter 2, Morgan Kaufmann, 2003