Constraint Satisfaction Problems

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Exercise Sheet 5

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Exercise 5.1 (2+2 Points)
Let $D = \{0, \dots, 5\}$, and N the constraint network defined by:

$$\Big\langle (v_1, v_2, v_3), (D, D, D), \Big\{ \big((v_1, v_2), \{ (x, y) \in D^2 \mid x + y = 3 \}\big), \\ \big((v_2, v_3), \{ (x, y) \in D^2 \mid x + y \leq 3\}\big), \big((v_1, v_3), \{ (x, y) \in D^2 \mid x \leq y \}\big)\Big\} \Big\rangle$$

(a) Apply the AC3 algorithm to N. Provide the state of the queue and the result of Revise (i.e., the resulting state of the current D_i) once for every iteration of the while loop.

(b) Apply the PC2 algorithm to N. Provide the state of the queue and the result of Revise-3 once for every iteration of the while loop.

Exercise 5.2 (2 Points)
Let N be a satisfiable constraint network and N_0 its (non-binary) minimal network. Prove that N_0 is globally consistent.

Exercise 5.3 (1+1+2 Points)
Let N be a binary, path-consistent constraint network with variables $V = (v_1, \dots, v_n)$ over the same Boolean domain $D_i = \{0, 1\}, 1 \leq i \leq n$. Further, let all constraints be non-empty, i.e., $R_{(v_i, v_j)} \neq \emptyset$ for all $v_i, v_j \in V$.
We prove that N is always satisfiable by induction. This exercise is split into three parts: Part (a) is our base case and Part (c) is our induction step which relies on Part (b) as a helpful property.

(a) Prove that for every pair of distinct variables $v_i, v_j \in V$ there exists a consistent (partial) assignment for v_i, v_j.

(b) Let $l > 1$ and $A_1, \dots, A_l$ be non-empty sets with $A_m \subseteq \{0, 1\}, 1 \leq m \leq l$. Prove that if $\bigcap_{1 \leq m \leq l} A_m = \emptyset$, then there exist $1 \leq o \neq p \leq l$ such that $A_o \cap A_p = \emptyset$.

(c) Prove that a consistent (partial) assignment on $v_1, \dots, v_{k-1}, 2 < k \leq n$ can be extended to v_k.
Hint: You can construct a proof by contradiction: Assume that the partial assignment $(a_1, \dots, a_{k-1})$ cannot be extended to v_k, i.e.,
$\bigcap_{2 \leq i \leq k-1} \{a_k \mid (a_i, a_k) \in R_{(v_i, v_k)}\} = \emptyset$ and apply (b).

Thus, we have shown that path consistency decides satisfiability of binary constraint networks over a Boolean domain. Such constraint problems are tractable.