

Foundations of Programming Languages and Software Engineering

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Overview

- Basics
 - Relations
 - Induction

- Terms and All That
 - Syntax
 - Semantics

Definition

- A **binary relation** on sets M_1 and M_2 is a set $R \subseteq M_1 \times M_2$ of pairs of elements from M_1 and M_2 , respectively. If $M_1 = M_2 = M$, we simply call R a binary relation on M .
- We say that $m_1 \in M_1$ and $m_2 \in M_2$ are **related by** R iff $(m_1, m_2) \in R$.
- We often write $m_1 R m_2$ instead of $(m_1, m_2) \in R$.

Properties of Binary Relations (1)

Definition

Let R be a binary relation on M .

- R is **reflexive** iff $m R m$ for all $m \in M$.
- R is **symmetric** iff $m R m'$ implies $m' R m$.
- R is **transitive** iff $m_1 R m_2$ and $m_2 R m_3$ imply $m_1 R m_3$.
- R is an **equivalence relation** iff it is reflexive, symmetric, and transitive.

Properties of Binary Relations (2)

Definition

Let R be a binary relation on M .

- The **reflexive closure** of R is the smallest reflexive relation R' such that $R \subseteq R'$.
- The **transitive closure** of R is the smallest transitive relation R' such that $R \subseteq R'$. It is often written R^+ .
- The **reflexive and transitive closure** of R is the smallest reflexive and transitive relation R' such that $R \subseteq R'$. It is often written R^* .

Induction Principles

Suppose P is some property on natural numbers.

Principle of ordinary induction on natural numbers

If $P(0)$

and, for all $i \in \mathbb{N}$, $P(i)$ implies $P(i + 1)$,

then $P(n)$ holds for all $n \in \mathbb{N}$.

Base case

Induction step

Conclusion

The assumption “ $P(i)$ ” in the induction step is called the **induction hypothesis** (IH for short).

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Principle of complete induction on natural numbers

If, for each $n \in \mathbb{N}$,
given $P(i)$ for all $i < n$
we can show $P(n)$,
then $P(n)$ holds for all $n \in \mathbb{N}$.

Example

Lemma

For all $n \in \mathbb{N}$, $\sum_{i=1}^n (2i - 1) = n^2$.

Proof. The proof is by ordinary induction on n .

- If $n = 0$, then both sides of the equation are 0.
- Suppose the lemma holds for some $k \in \mathbb{N}$. We then have:

$$\begin{aligned}\sum_{i=1}^{k+1} (2i - 1) &= \sum_{i=1}^k (2i - 1) + (2(k + 1) - 1) \\ &\stackrel{\text{(IH)}}{=} k^2 + 2k + 1 \\ &= (k + 1)^2 \quad \square\end{aligned}$$

Definition

- A **signature** Σ is a set of **function symbols**, where each $f \in \Sigma$ is associated with a natural number n called the **arity** of f .
- $\Sigma^{(n)}$ denotes the set of all n -ary elements of Σ .
- The elements of $\Sigma^{(0)}$ are also called **constant symbols**.

Example

Signature Σ_{prop} for propositional logic

$$\Sigma_{prop} = \{\mathbf{T}^{(0)}, \mathbf{F}^{(0)}, \neg^{(1)}, \wedge^{(2)}, \vee^{(2)}\}$$

$$\Sigma_{prop}^{(0)} = \{\mathbf{T}, \mathbf{F}\}$$

$$\Sigma_{prop}^{(1)} = \{\neg\}$$

$$\Sigma_{prop}^{(2)} = \{\wedge, \vee\}$$

Definition

Let Σ be a signature and X a set of **variables** such that $\Sigma \cap X = \emptyset$. The set $T(\Sigma, X)$ of all Σ -**terms** over X is inductively defined as

- $X \subseteq T(\Sigma, X)$,
- for all $n \in \mathbb{N}$, all $f \in \Sigma^{(n)}$, and all $t_1, \dots, t_n \in T(\Sigma, X)$, we have $f(t_1, \dots, t_n) \in T(\Sigma, X)$

Note:

- For a constant symbol $f \in \Sigma^{(0)}$, we often write the term $f()$ as f .
- From now on, we leave the variable set $X = \{x, x_1, x_2, \dots, y, y_1, y_2, \dots, z, z_1, z_2, \dots\}$ implicit

Example

Suppose $\Sigma = \Sigma_{prop}$. Then

$$\vee(\neg(x_{42}), \wedge(\mathbf{T}, x_3)) \in T(\Sigma, X)$$

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Alternative notation

Infix notation (with implicit operator precedence order):

$$\neg x_{42} \vee \mathbf{T} \wedge x_3$$

Unique Decomposition of Terms

- In our current view, **equality of terms** means **syntactic equality**.
- Therefore, if $t, s \in T(\Sigma, X)$ and $t = f(t_1, \dots, t_n)$ and $s = g(s_1, \dots, s_m)$, and $t = s$, then $f = g$, $n = m$, and $t_i = s_i$ for all $i \in \{1, \dots, n\}$.

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- Later, we consider a kind of **semantic** equality: $+(1, 3)$ might be equal to $+(2, 2)$.

Positions and Size of Terms

Definition

Suppose $t \in T(\Sigma, X)$.

- The set of **positions** of term t is a set $Pos(t)$ of strings over the alphabet of natural numbers. It is inductively defined as follows:
 - If $t = x \in X$, then $Pos(t) := \{\epsilon\}$
 - If $t = f(t_1, \dots, t_n)$, then

$$Pos(t) := \{\epsilon\} \cup \bigcup_{i=1}^n \{ip \mid p \in Pos(t_i)\}$$

- The position ϵ is called the **root position** of t , the function or variable at this position is called the **root symbol** of t .
- The **size** $|t|$ of t is the cardinality of $Pos(t)$.

Subterms and Replacing

Definition (Subterm)

For $p \in Pos(t)$, the **subterm** of t at position p , denoted by $t|_p$, is defined by induction on the length of p :

$$\begin{aligned}t|_\epsilon &:= t \\ f(t_1, \dots, t_n)|_{ip} &:= t_i|_p\end{aligned}$$

($ip \in Pos(t)$ implies that $t = f(t_1, \dots, t_n)$ with $0 \leq i \leq n$.)

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Definition (Replacing)

For $p \in Pos(t)$, we denote by $t[s]_p$ the term that is obtained from t by replacing the subterm at position p by s , i.e.

$$\begin{aligned}t[s]_\epsilon &:= s \\ f(t_1, \dots, t_n)[s]_{ip} &:= f(t_1, \dots, t_i[s]_p, \dots, t_n)\end{aligned}$$

Examples

Suppose $t = \vee(\neg(x_{42}), \wedge(\mathbf{T}, x_3))$

- $Pos(t) = \{\epsilon, 1, 12, 2, 21, 22\}$
- $|t| = 6$ (number of nodes in the tree)
- $t|_2 = \wedge(\mathbf{T}, x_3)$
- $t[\neg(\mathbf{F})]|_2 = \vee(\neg(x_{42}), \neg(\mathbf{F}))$

An Induction Principle for Terms

Term Induction

To prove that a property P holds for all $t \in T(\Sigma, X)$, we have to show the following properties:

- **Base case**

$P(x)$ holds for all $x \in X$ and $P(f)$ holds for all $f \in \Sigma^{(0)}$.

- **Induction step**

Suppose $n > 0$, $f \in \Sigma^{(n)}$, and $t_1, \dots, t_n \in T(\Sigma, X)$.

Then $P(f(t_1, \dots, t_n))$ holds assuming $P(t_1), \dots, P(t_n)$.

Example for Term Induction

Lemma

For all terms t , the set $Pos(t)$ is prefix closed, i.e. if $wv \in Pos(t)$ then $w \in Pos(t)$.

Definition

Let Σ be a signature.

- A $T(\Sigma, X)$ -substitution is a function $\sigma : X \rightarrow T(\Sigma, X)$ such that $\sigma(x) \neq x$ for only finitely many x s.
- The domain of σ is $Dom(\sigma) := \{x \in X \mid \sigma(x) \neq x\}$.
- We write $\{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$ for a substitution that maps x_i to t_i and has domain $Dom(\sigma) = \{x_1, \dots, x_n\}$.

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- We write $\{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$ for a substitution that maps x_i to t_i and has domain $Dom(\sigma) = \{x_1, \dots, x_n\}$.
- A $T(\Sigma, X)$ -substitution σ is extended to a mapping $\sigma : T(\Sigma, X) \rightarrow T(\Sigma, X)$ on arbitrary terms as follows:
$$\sigma(f(t_1, \dots, t_n)) := f(\sigma(t_1), \dots, \sigma(t_n))$$

Substitutions. Explanation

Note

Applying the extension of a substitution σ to a term **simultaneously** replaces all occurrences of a variable by their respective σ -image.

Example

A substitution on terms from $T(\Sigma_{prop}, X)$

$$\Sigma = \Sigma_{prop}$$

$$\sigma = \{x \mapsto \neg z, y \mapsto x \vee \mathbf{F}\}$$

$$t = x \vee y \wedge z$$

$$\sigma(t) = \neg z \vee (x \vee \mathbf{F}) \wedge z$$

Composing Substitutions

Definition

The **composition** $\sigma\tau$ of two substitutions σ and τ is defined as $\sigma\tau(x) := \sigma(\tau(x))$.

Lemma

Composition of substitutions is an associative operation where the identity substitution is the unit.

Definition

Let Σ be a signature. A Σ -algebra $\mathcal{A} = (A, \mathcal{J})$ consists of

- a **carrier set** A , and
- an **interpretation function** \mathcal{J} that associates with each function symbol $f \in \Sigma^{(n)}$ a function $\mathcal{J}(f) : A^n \rightarrow A$.

The Σ_{prop} -Algebra \mathcal{A}_{prop}

$$\mathcal{A}_{prop} = (\mathbf{A}_{prop}, \mathcal{J}_{prop})$$

$$\mathbf{A}_{prop} = \{0, 1\}$$

$$\mathcal{J}_{prop}(\mathbf{F}) = 0$$

$$\mathcal{J}_{prop}(\mathbf{T}) = 1$$

$$\mathcal{J}_{prop}(\neg)(x) = 1 - x$$

$$\mathcal{J}_{prop}(\vee)(x, y) = \max(x, y)$$

$$\mathcal{J}_{prop}(\wedge)(x, y) = \min(x, y)$$

Definition

Let $\mathcal{A} = (A, \mathcal{J})$ be a Σ -algebra.

- A **variable assignment** is a function $\alpha : X \rightarrow A$ that assigns every variable a value in the carrier set.
- Given a variable assignment α , the **interpretation function** \mathcal{J} is extended to a function on terms,

$\mathcal{J}_\alpha : T(\Sigma, X) \rightarrow A$, as follows:

$$\mathcal{J}_\alpha(x) = \alpha(x) \quad (x \in X)$$

$$\mathcal{J}_\alpha(f(t_1, \dots, t_n)) = \mathcal{J}(f)(\mathcal{J}_\alpha(t_1), \dots, \mathcal{J}_\alpha(t_n))$$

- The restriction of \mathcal{J}_α to variable free-terms, $\mathcal{J}_\alpha : T(\Sigma, \emptyset) \rightarrow A$, is usually denoted by \mathcal{J} since the α does not matter.

Example

Interpretation of $\vee(\neg(x_{42}), \wedge(\mathbf{T}, x_3)) \in T(\Sigma_{prop}, X)$

Suppose $\alpha : X \rightarrow A_{prop}$ is a function such that

$$\alpha(x_{42}) = 0$$

$$\alpha(x_3) = 1$$

Then we have

$$\begin{aligned} \mathcal{J}_\alpha(\vee(\neg(x_{42}), \wedge(\mathbf{T}, x_3))) &= \mathcal{J}(\vee)(\mathcal{J}_\alpha(\neg(x_{42})), \mathcal{J}_\alpha(\wedge(\mathbf{T}, x_3))) \\ &= \max(\mathcal{J}(\neg)(\mathcal{J}_\alpha(x_{42})), \\ &\quad \mathcal{J}(\wedge)(\mathcal{J}_\alpha(\mathbf{T}), \mathcal{J}_\alpha(x_3))) \\ &= \max(1 - \alpha(x_{42}), \min(\mathcal{J}(\mathbf{T}), \alpha(x_3))) \\ &= \max(1 - 0, \min(1, 1)) = 1 \end{aligned}$$