12 Edit distance

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Problem: similarity of strings

Edit distance

For two strings *A* and *B*, compute, as efficiently as possible, the edit distance *D(A,B)* and a minimal sequence of edit operations which transforms *A* into *B*.

i n f - - - o r m a t i k -
i n t e r p o l - a t i o n

Application of edit distance

Approximate string matching

For a given text *T*, a pattern *P*, and a distance *d*, find all substrings *P´* in *T* with $D(P,P') \leq d$

Sequence alignment

Find optimal alignments of DNA sequences

G A G C A - C T T G G A T T C T C G G
- - - C A C G T G G - - - - - - - - - -

Edit distance

Given: two strings $A = a_1 a_2 \ldots a_m$ and $B = b_1 b_2 \ldots b_n$

Goal: find minimal cost $D(A,B)$ for a sequence of edit operations to transform *A* into *B*.

Edit operations:

1. Replace a character in *A* by a character from *B*
2. Delete a character from *A*
3. Insert a character into *A*

Edit distance

Cost model:

$$c(a,b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

$a = \varepsilon,\ b = \varepsilon$ possible

We assume the triangle inequality holds for *c*:

$$c(a,c) \leq c(a,b) + c(b,c)$$

⟶ Each character is changed at most once

Edit distances

Trace as representation of edit sequences

```
A =   b a a c a a b c
      | |  /  /  |  /
B =   a b a c b c a c
```

or using indents

```
A = - b a a c a - a b c
      | |   |  |   |   |
B = a b a - c b c a - c
```

Edit distance (cost): 5

An optimal trace can be divided into two optimal subtraces.

Computation of the edit distance

Let $A_i = a_1 \ldots a_i$ and $B_j = b_1 \ldots b_j$

$D_{i,j} = D(A_i, B_j)$

Computations of the edit distances

- Three possibilities of ending a trace:
- 1. a_m is replaced by b_n :
  $D_{m,n} = D_{m-1,n-1} + c(a_m, b_n)$
  ($c(a_m, b_n)$ can be 0 or 1)

  $a_1,\ldots,a_{m-1},a_m$
  $b_1,\ldots,b_{n-1},b_n$
- 2. a_m is deleted: $D_{m,n} = D_{m-1,n} + 1$
- 3. b_n is inserted: $D_{m,n} = D_{m,n-1} + 1$

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  $b_1,\ldots,b_n$
- 3. b_n is inserted: $D_{m,n} = D_{m,n-1} + 1$

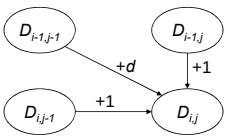
  $a_1,\ldots,a_m$
  $b_1,\ldots,b_{n-1}\,b_n$

Computation of the edit distance

- Recurrence relation, if $m,n \geq 1$:

$$D_{m,n} = \min \begin{cases} D_{m-1,n-1} + c(a_m, b_n), \\ D_{m-1,n} + 1, \\ D_{m,n-1} + 1 \end{cases}$$

- ⟶ Computation of all $D_{i,j}$ is required, $0 \leq i \leq m$, $0 \leq j \leq n$.



Recurrence relation for the edit distance

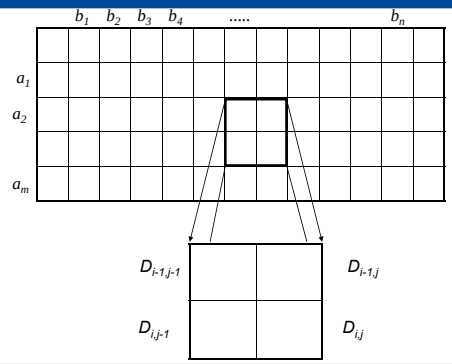
Base cases:

$D_{0,0} = D(\varepsilon, \varepsilon) = 0$
$D_{0,j} = D(\varepsilon, B_j) = j$
$D_{i,0} = D(A_i, \varepsilon) = i$

Recurrence equation:

$$D_{i,j} = \min \begin{cases} D_{i-1,j-1} + c(a_i, b_j) \\ D_{i-1,j} + 1, \\ D_{i,j-1} + 1 \end{cases}$$

Order of computation for the edit distance



Algorithm for the edit distance

Algorithm edit_distance
Input: two strings $A = a_1 .... a_m$ and $B = b_1 ... b_n$
Output: the matrix $D = (D_{ij})$

```
1 D[0,0] := 0
2 for i := 1 to m do D[i,0] = i
3 for j := 1 to n do D[0,j] = j
4 for i := 1 to m do
5   for j := 1 to n do
6       D[i,j] := min( D[i - 1,j] + 1,
7                      D[i,j - 1] + 1,
8                      D[i –1, j – 1] + c(a_i,b_j))
```


Example

		a	b	a	c
	0	1	2	3	4
b	1				
a	2				
a	3				
c	4				

Example

	j	0	1	2	3	4
			a	b	a	c
i						
0		0	1	2	3	4
1	b	1				
2	a	2				
3	a	3				
4	c	4				

Example

	j	0	1	2	3	4
			a	b	a	c
i						
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}				
2	a	2_{del}				
3	a	3_{del}				
4	c	4_{del}				

Example

	j	0	1	2	3	4
			a	b	a	c
i						
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1			
2	a	2_{del}				
3	a	3_{del}				
4	c	4_{del}				

Example

i	j	0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1			
2	a	2_{del}				
3	a	3_{del}				
4	c	4_{del}				

D(ε , 2) +1 = 3

D(1 , 1) +1 = 2

D(ε , 1) +0 = 1

Example

i	j	0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1			
2	a	2_{del}				
3	a	3_{del}				
4	c	4_{del}				

D(ε , 2) +1 = 3

D(1 , 1) +1 = 2

D(ε , 1) +0 = 1

Example

i	j	0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1	1		
2	a	2_{del}				
3	a	3_{del}				
4	c	4_{del}				

Example

i	j	0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1	1	2	3
2	a	2_{del}	1			
3	a	3_{del}	2			
4	c	4_{del}	3			

Example

i	j	0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1	1	2	3
2	a	2_{del}	1	2		
3	a	3_{del}	2			
4	c	4_{del}	3			

Example

i	j	0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1	1	2	3
2	a	2_{del}	1	2	1	2
3	a	3_{del}	2	2		
4	c	4_{del}	3	3		

Example

i \ j		0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1	1	2	3
2	a	2_{del}	1	2	1	2
3	a	3_{del}	2	2	2	2
4	c	4_{del}	3	3	3	

Example

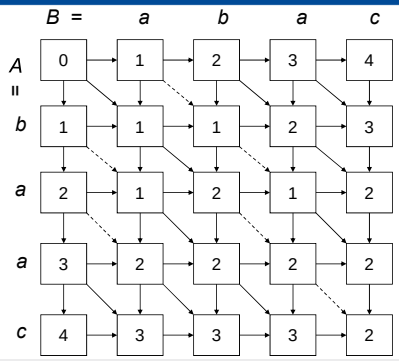
i \ j		0	1	2	3	4
			a	b	a	c
0		0	1_{ins}	2_{ins}	3_{ins}	4_{ins}
1	b	1_{del}	1	1	2	3
2	a	2_{del}	1	2	1	2
3	a	3_{del}	2	2	2	2
4	c	4_{del}	3	3	3	2

Computation of the edit operations

Algorithm edit_operations (*i,j*)
Input: matrix *D* (computed)
1 **if** *i* = 0 **and** j = 0 **then return**
2 **if** *i* ≠ 0 **and** *D*[*i,j*] = *D*[*i* – 1 , *j*] + 1
3 **then** „delete *a*[*i*]“
4 edit_operations (*i* – 1, *j*)
5 **else if** *j* ≠ 0 and *D*[*i,j*] = *D*[*i*, *j* – 1] + 1
6 **then** „insert *b*[*j*]“
7 edit_operations (*i*, *j* – 1)
8 **else**
/ D[i,j] = D[i – 1, j – 1] + c(a[i], b[j]) */*
9 „replace *a[i]* by *b[j]* “
10 edit_operations (*i* – 1, *j* – 1)

Initial call: edit_operations(*m*,*n*)

Trace graph of the edit operations



Sub-graph of the edit operations

Trace graph: All possible traces which transform A into B, directed edges from vertex (*i, j*) to (*i* + 1, *j*), (*i*, *j* + 1) and (*i* + 1, *j* + 1).

Weights of the edges represent the edit costs.

Costs are monotonically increasing along an optimal path.

Each path from the upper left corner to the lower right corner, with monotonically increasing costs, represents an optimal trace.

Dynamic programming

- The algorithm just presented is an example of dynamic programming, an algorithm design technique often used for optimization problems.
- Generally usable for recursive approaches if the same partial solutions are required more than once
- Approach: store partial results in a table
- Advantage: better time complexity, often polynomial instead of exponential