# 10 Randomized algorithms

Summer Term 2011

Jan-Georg Smaus

Albert-Ludwigs-Universität Freiburg

## Randomized algorithms

Overview

- Classes of randomized algorithms
- Quicksort
- Randomized Quicksort
- Randomized primality test
- Cryptography



## 1. Classes of randomized algorithms

- **Las Vegas** algorithms
  always correct; expected running time "probably fast"

  Example: randomized Quicksort

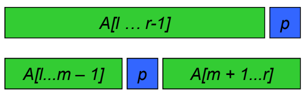
- **Monte Carlo** algorithms (mostly correct):
  probably correct; guaranteed running time

  Example: randomized primality test



## 2. Quicksort

Unsorted range *A*[*l*, *r*] in array *A*




**Quicksort** **Quicksort**



## Quicksort

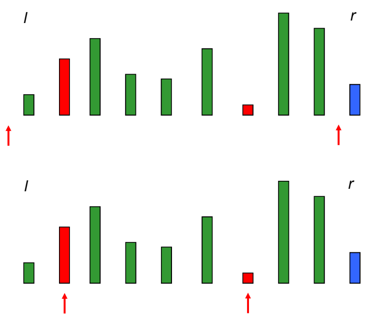
**Algorithm:** *Quicksort*

**Input:** unsorted range [*l*, *r*] in array *A*
**Output:** sorted range [*l*, *r*] in array *A*

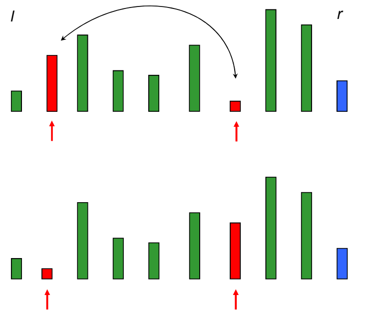
```
if r > l
  then choose pivot element p = A[r]
    m = divide(A, l , r)
      /* Divide A according to p:
         A[l],....,A[m – 1] ≤ p ≤ A[m + 1],...,A[r]
      */
Quicksort(A, l , m - 1)
Quicksort (A, m + 1, r)
```



## The *divide* step

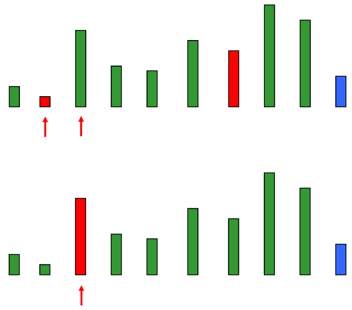


## The *divide* step



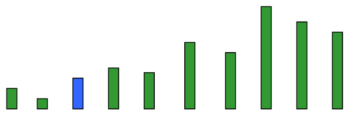



## The *divide* step





## The *divide* step

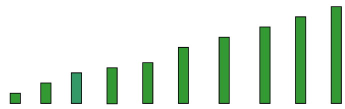


*divide(A, l , r)*:

- returns the index of the pivot element in *A*
- can be done in time $O(r – l)$



## Worst case input



*n* elements:

Running time: $(n-1) + (n-2) + \ldots + 2 + 1 = n\cdot(n-1)/2 = \Omega(n^2)$



## 3. Randomized Quicksort

**Algorithm:** Quicksort
**Input:** unsorted range [*l*, *r*] in array *A*
**Output:** sorted range [*l*, *r*] in array *A*

```
if  r > l
    then randomly choose a pivot element p = A[i] in range [l, r]
         swap A[i] and A[r]
         m = divide(A, l, r)
         /* Divide A according to p:
              A[l],....,A[m – 1] ≤ p ≤ A[m + 1],...,A[r]
          */
          Quicksort(A, l, m - 1)
          Quicksort(A, m + 1, r)
```



## Analysis 1

*n* elements; let $S_i$ be the *i*-th smallest element

- $S_1$ is chosen as pivot with probability $1/n$:
- Sub-problems of sizes $0$ and $n-1$

•
•
•

- $S_k$ is chosen as pivot with probability $1/n$:
- Sub-problems of sizes $k-1$ and $n-k$

•
•
•

- $S_n$ is chosen as pivot with probability $1/n$:
- Sub-problems of sizes $n-1$ and $0$

## Analysis 1

**Expected running time**:

$$E(T(n)) = \frac{1}{n}\sum_{k=0}^{n-1}(E(T(k)) + E(T(n-k-1))) + \Theta(n)$$

$$= \frac{2}{n}\sum_{k=0}^{n-1}E(T(k)) + \Theta(n)$$

($\Theta(n)$ *because* of the divide-procedure)

One can show that $E(T(n)) \in O(n \log n)$.

With high probability, bad inputs cannot spoil the performance.



## 4. Primality test

**Definition:**
An integer $p \geq 2$ is prime **iff** ($a \mid p \rightarrow a = 1$ **or** $a = p$).

**Algorithm:** deterministic primality test (naive)
**Input:** integer $n \geq 2$
**Output:** answer to the question: Is $n$ prime?

**if** $n = 2$ **then return** true
**if** $n$ even **then return** false
**for** $i = 1$ **to** $\sqrt{n/2}$ **do**
  **if** $2i + 1$ divides $n$
  **then return** false
**return** true

Complexity: $\Theta(\sqrt{n})$ where $n$ is the input, but the input size is just $\log n$.



## Primality test

**Goal:**
Randomized method
- Polynomial time complexity (in the length of the input)
- If answer is "not prime", then $n$ is not prime
- If answer is "prime", then the probability that $n$ is not prime is at most $p>0$

$k$ iterations: probability that $n$ is not prime is at most $p^k$.



## Randomized primality test

**Theorem:** (Fermat's little theorem)
If $p$ prime and $0 < a < p$, then
$a^{p-1} \bmod p = 1$. (I.e., $p$ divides $a^{p-1} - 1$)

**Definition:**
$n$ is pseudoprime to base $a$, if $n$ not prime and
$a^{n-1} \bmod n = 1$.

**Example:** $n = 11 * 31 = 341$, $a = 2$
$2^{340} \bmod 341 = 1$

**but:** $n = 341$, $a = 3$
$3^{340} \bmod 341 = 56 \neq 1$



## Randomized primality test 1

1 Randomly choose $a \in [2, n-1]$
2 **if** $a^{n-1} \bmod n = 1$
3 **then** $n$ is possibly prime
4 **else** $n$ is definitely not prime

Advantage: This only takes polynomial time.

**Examples:** $n = 17$, $a = 2$: $2^{16} = 65536$. $65536 \bmod 17 = 1$.
$n = 23$, $2^{22} = 4194304$. $4194394 \bmod 23 = 1$.
$n = 341$, $2^{340} \bmod 341 = 1$.
17 and 23 are indeed prime, 341 is not!

*Prob*($n$ is not prim, but $a^{n-1} \bmod n = 1$ ) ?



## Carmichael numbers

**Problem:** Carmichael numbers

**Definition:** An integer $n$ is called Carmichael number if
$a^{n-1} \bmod n = 1$
for all $a$ with $\mathrm{GCD}(a, n) = 1$. **(GCD = greatest common divisor)**

**Example:**
Smallest Carmichael number: $561 = 3 * 11 * 17$

561 is pseudoprime to any base $a$ that is not divisible by 3 or 11 or 17.
To show that 561 is not prime, we hence need a base $a$ that is divisible by 3 or 11 or 17. This is still quite likely to find, but there are worse examples.

# Randomized primality test 2

**Theorem:**

If $p$ prime and $0 < a < p$, then the only solutions to the equation

$a^2 \bmod p = 1$

are $a = 1$ and $a = p - 1$.

**Definition:**

$a$ is called non-trivial square root of 1 mod $n$, if

$a^2 \bmod n = 1$ and $a \neq 1, n - 1$.

**Example:** $n = 35$

$6^2 \bmod 35 = 1$



# Fast exponentiation

**Idea:**

During the computation of $a^{n-1}$ ($0 < a < n$ randomly chosen), which we need for the first primality test, test as a byproduct whether there is a non-trivial square root 1 mod $n$.

**Method for the computation of $a^n$:**

**Case 1**: [$n$ is even]

$a^n = a^{n/2} * a^{n/2}$

**Case 2**: [$n$ is odd]

$a^n = a^{(n-1)/2} * a^{(n-1)/2} * a$



# Fast exponentiation

**Example:**

$a^{62} = (a^{31})^2$
$a^{31} = (a^{15})^2 * a$
$a^{15} = (a^7)^2 * a$
$a^7 = (a^3)^2 * a$
$a^3 = (a)^2 * a$

To compute $a^n$, the exponents are obviously divided by 2 (at least) in each step. Hence there are O(log $n$) intermediate steps.

In each intermediate step, we multiply and compute the square for operands of number size $O(a^n)$ and hence representation size $O(\log a^n)$, leading to $O(\log^2 a^n)$ for each intermediate step.

Overall complexity: $O(\log^2 a^n \log n)$



# Fast exponentiation + squares

```
boolean isProbablyPrime;

power(int a, int p, int n) {
    /* computes a^p mod n and checks during the
       computation whether there is an x with
       x² mod n = 1 and x ≠ 1, n-1 */

    if (p == 0) return 1;
    x = power(a, p/2, n)
    result = (x * x) % n;
```



# Fast exponentiation + squares

```
/* check whether x² mod n = 1 and x ≠ 1, n-1 */
    if (result == 1 && x != 1 && x != n -1 )
        isProbablyPrime = false;

    if (p % 2 == 1)
        result = (a * result) % n;

    return result;
}
```

Complexity: O($\log^2 n \log p$)



# Combined Procedure Miller-Rabin

```
primalityTest(int n) {
    /* carries out the randomized primality test for
       a randomly selected a */

    a = random(2, n-1);

    isProbablyPrime = true;

    result = power(a, n-1, n);

    if (result != 1 || !isProbablyPrime)
        return false;
    else
        return true;
}
```

## Combined Procedure Miller-Rabin

**Theorem:**

If $n$ is not prime, there are at most $\frac{n-9}{4}$

integers $0 < a < n$, for which the algorithm primalityTest fails. Hence the probability of failure is

$$\frac{\frac{n-9}{4}}{n} < \frac{\frac{n}{4}}{n} = \frac{1}{4}$$

If for a number $n$ we do $\log n$ tests we get a probability of

$$\left(\frac{1}{4}\right)^{\log n} = \frac{1}{n^2}$$

Of failure. E.g. we might take $n$ around $2^{500}$.

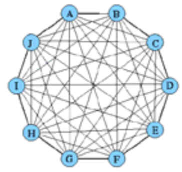


## Application: cryptosystems

**Traditional encryption of messages with secret keys**

**Disadvantages:**

1. The key $k$ has to be exchanged between A and B before the transmission of the message.
2. For messages between $n$ parties $n(n-1)/2$ keys are required.




**Advantage:**

Encryption and decryption can be computed very efficiently.



## Desired properties of cryptographic systems

- confidential transmission
- integrity of data
- authenticity of the sender
- reliable transmission



## Public-key cryptosystems

Diffie and Hellman (1976)

**Idea:** Each participant A has two keys:

1. a public key $P_A$ accessible to every other participant
2. a private (or: secret) key $S_A$ only known to $A$.



## Public-key cryptosystems

$D$ = set of all legal messages, e.g. the set of all bit strings of finite length

$$P_A, S_A : D \rightarrow D$$

**Three conditions:**

1. $P_A$ and $S_A$ can be computed efficiently

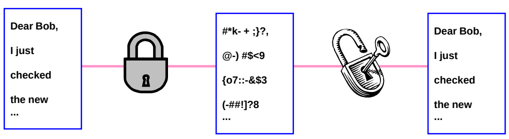
2. $S_A(P_A(M)) = M$ and $P_A(S_A(M)) = M$
($P_A$ is the inverse function of $S_A$ and vice-versa)

3. $S_A$ cannot be computed from $P_A$ with reasonable effort.



## Encryption in a public-key cryptosystem

$A$ sends a message $M$ to $B$.



## Encryption in a public-key cryptosystem

1. ***A*** accesses ***B***'s public key $P_B$ (from a public directory or directly from *B*).
2. ***A*** computes the encrypted message $C = P_B(M)$ and sends *C to* ***B***.
3. After ***B*** has received message *C*, ***B*** decrypts the message with his own private key $S_B$: $M = S_B(C)$



## Generating a digital signature

***A*** sends a digitally signed message $M'$ to ***B***:

1. ***A*** computes the digital signature $\sigma$ for $M'$ with her own private key:
   $\sigma = S_A(M')$
2. ***A*** sends the pair $(M',\sigma)$ to ***B***.
3. After receiving $(M',\sigma)$, ***B*** verifies the digital signature:

$$P_A(\sigma) = M'$$

$\sigma$ can by verified by anybody via the public $P_A$.



## RSA cryptosystems

R. Rivest, A. Shamir, L. Adleman

Generating the public and private keys:

1. Randomly select two primes $p$ and $q$ of similar size, each with $l+1$ bits ($l \geq 500$).
2. Let $n = p \cdot q$
3. Let $e$ be a (small) integer that does not divide $(p - 1)\cdot(q - 1)$.
4. Calculate $d = e^{-1} \bmod (p - 1)(q - 1)$

   i.e.:

$$d \cdot e \equiv 1 \bmod (p - 1)(q - 1)$$



## RSA cryptosystems

5. Publish $P = (e, n)$ as public key
6. Keep $S = (d, p, q)$ as private key

Divide message (described in a binary string) in blocks of size $2 \cdot l$.
Interpret each block $M$ as a binary number: $0 \leq M < 2^{2 \cdot l}$

$$P(M) = M^e \bmod n \qquad S(M) = M^d \bmod n$$



## RSA has the desired properties ...

We have to show that

1. $P_A$ and $S_A$ can be computed efficiently.
2. $S_A(P_A(M)) = M$ and $P_A(S_A(M)) = M$
   ($P_A$ is the inverse function of $S_A$ and vice-versa)
3. $S_A$ cannot be computed from $P_A$ with reasonable effort.

1 is fulfilled because exponentiation can be computed efficiently.



## P and S are inverses

We have (some basic math ...)

$$\begin{aligned} M^{(p-1)\cdot(q-1)} &\equiv 1 && \bmod p \\ M^{(p-1)\cdot(q-1)} &\equiv 1 && \bmod q \\ M^{(p-1)\cdot(q-1)} &\equiv 1 && \bmod p \cdot q \end{aligned}$$

and hence

$$\begin{aligned} S(P(M)) &\equiv (M^e)^d && \bmod n \\ &\equiv M^{e \cdot d} && \bmod n \\ &\equiv M^{1+r\cdot(p-1)\cdot(q-1)} && \bmod n \\ &\equiv M \cdot (M^{(p-1)\cdot(q-1)})^r && \bmod n \\ &\equiv M && \bmod n \end{aligned}$$

The other direction is analogous.

## S is hard to compute

This is unproven!

According to current knowledge, to compute *d* from e one would need to know *p* and *q*.

Also according to current knowledge, computing *p* and *q* from *n* is hard.

Even the fastest computers have never cracked RSA!



## Summary

- We have seen two randomised algorithms:
  - Quicksort
  - Prime test
- We have also seen an application of big prime numbers: cryptography.