# 9 Dynamic tables

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## Dynamic tables

Problem:
Maintenance of a table under the operations insert and delete such that
- the table size can be adjusted to the number of elements
- a fixed portion of the table is always filled with elements
- the costs for *n* insert or delete operations are in *O*(*n*).

Organisation of the table: hash table, heap, stack, etc.

Load factor $\alpha_T$ : fraction of table spaces of *T* which are occupied.

Cost model:

Insertion or deletion of an element causes cost 1, if the table is not filled yet.
If the table size is changed, all elements must be copied.



## Initialisation

```
class dynamicTable {
       private int [] table;
       private int size;
       private int num;
       dynamicTable () {
           table = new int [1];       // initialize empty table
           size = 1;
           num = 0;
       }
```



## Expansion strategy: insert

Double the table size whenever an element is inserted in the fully occupied table!

```
public void insert (int x) {
       if (num == size) {
           int[] newTable = new int[2*size];
           for (int i=0; i < size; i++)
               insert table[i] in newTable;
           table = newTable;
           size = 2*size;
       }
       insert x in table;
       num = num + 1;
   }
```



## *Insert* operation in an initially empty table

$t_i$ = cost of the *i*-th insert operation

**Worst case:**

$t_i = 1$, if the table was not full before operation *i*
$t_i = (i - 1) + 1$, if the table was full before operation *i*
Hence, *n* insert operations require costs of at most

$$\sum_{i=1}^{n}(i) = O(n^2)$$

**Amortized worst case:**
Aggregate analysis, accounting method, potential method



## Amortized cost

- Let the real cost of the *i*th insertion be $t_i$. We want to define the amortized cost $a_i$ such that

$$\sum a_i \geq \sum t_i \in O(n).$$

## Potential method

$T$ table with

- $k = T.num$ elements and
- $s = T.size$ spaces

**Potential function**

$$\phi(T) = 2\,k - s$$

We also write $\phi_i$ *for* $\phi(T)$ after the $i$th insert operation.

We now define $a_i = \phi_i - \phi_{i-1} + t_i$.



## Properties of the potential function

**Properties**

- $\phi_0 = \phi(T_0) = \phi$ (empty table) = -1
- For all $i \ge 1$ : $\phi_i = \phi(T_i) \ge 0$
  Since $\phi_n - \phi_0 \ge 0$, $\Sigma\, a_i$ is an upper bound for $\Sigma\, t_i$
- Directly before an expansion, $k = s$,
  hence $\phi(T) = k = s$.
- Directly after an expansion, $k = s/2$,
  hence $\phi(T) = 2k - s = 0$.



## Amortized cost of insert (1)

$k_i$ = # elements in $T$ after the $i$-th operation

$s_i$ = table size of $T$ after the $i$-th operation

**Case 1**: [ $i$-th operation does not trigger an expansion]

$$a_i = \phi_i - \phi_{i-1} + t_i = (2k_i - s_i) - (2k_{i-1} - s_{i-1}) + t_i =$$
$$2(k_i - k_{i-1}) - (s_i - s_{i-1}) + t_i =$$
$$2 - 0 + 1 \le 3.$$



## Amortized cost of insert (2)

**Case 2:** [ $i$-th operation triggers an expansion]

$$a_i = \phi_i - \phi_{i-1} + t_i = (2k_i - s_i) - (2k_{i-1} - s_{i-1}) + t_i =$$
$$2(k_i - k_{i-1}) - (s_i - s_{i-1}) + i =$$
$$2(1) - (2s_{i-1} - s_{i-1}) + i =$$
$$2 - s_{i-1} + i =$$
$$2 - (i - 1) + i \le 3.$$



## Insertion and deletion of elements

**Now:** contract table, if the load is too small!

**Goals:**

(1) Load factor is always bounded below by a constant

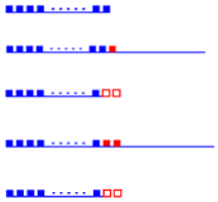
(2) Amortized cost of a single insert or delete operation is constant.

**First attempt:**

- Expansion: same as before
- Contraction: halve the table size as soon as table is less than ½ occupied (after the deletion)!



## „Bad" sequence of insert and delete operations



| | Cost |
|---|---|
| $n/2$ times insert (table fully occupied) | 3 $n/2$ |
| I: expansion | $n/2 + 1$ |
| D, D: contraction | $n/2 + 1$ |
| I, I : expansion | $n/2 + 1$ |
| D, D: contraction | |

Total cost of the sequence
$In/2,I,D,D,I,I,D,D,...$ of length $n$: $\Omega(n^2)$

## Second attempt

**Expansion:** (as before) double the table size, if an element is inserted in the full table.

**Contraction:** As soon as the load factor is below ¼, halve the table size.

**Hence:**

At least ¼ of the table is always occupied, i.e.

$$\frac{1}{4} \leq \alpha(T) \leq 1$$

Cost of a sequence of insert and delete operations?



## Analysis: insert and delete

$k$ = *T.num*, $s$ = *T.size*, $\alpha = k/s$

**Potential function** $\phi$

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$

Homework: Check $\sum a_i \geq \sum t_i$



## Analysis: insert and delete

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$

Directly after an expansion or contraction of the table:

$s = 2k$, hence $\phi(T) = 0$



## insert

$i$-th operation: $k_i = k_{i-1} + 1$

Case 1: $\alpha_{i-1} \geq ½$

Case 2: $\alpha_{i-1} < ½$

- Case 2.1: $\alpha_i < ½$
- Case 2.2: $\alpha_i \geq ½$



## insert

Case 2.1: $\alpha_{i-1} < ½$, $\alpha_i < ½$ (no expansion)

**Potential function** $\phi$

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$



## insert

Case 2.2: $\alpha_{i-1} < ½$, $\alpha_i \geq ½$ (no expansion)

**Potential function** $\phi$

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$

## delete

$k_i = k_{i-1} - 1$

Case 1: $\alpha_{i-1} < ½$

Case 1.1: deletion causes no contraction

$s_i = s_{i-1}$

**Potential function** $\phi$

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$



## delete

$k_i = k_{i-1} - 1$

Case 1: $\alpha_{i-1} < ½$

Case 1.2: $\alpha_{i-1} < ½$ deletion causes a contraction

$2s_i = s_{i-1}$

$k_{i-1} = s_{i-1}/4$

**Potential function** $\phi$

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$



## delete

Case 2: $\alpha_{i-1} \geq ½$ no contraction

$s_i = s_{i-1}$ $k_i = k_{i-1} - 1$

Case 2.1: $\alpha_{i-1} \geq ½$

**Potential function** $\phi$

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$



## delete

Case 2: $\alpha_{i-1} \geq ½$ no contraction

$s_i = s_{i-1}$ $k_i = k_{i-1} - 1$

Case 2.2: $\alpha_i < ½$

**Potential function** $\phi$

$$\phi(T) = \begin{cases} 2k - s, \text{ if } \alpha \geq 1/2 \\ s/2 - k, \text{ if } \alpha < 1/2 \end{cases}$$